Office of Naval Research
Contract Nonr-220(58)
NR-064-431

Technical Report No. 2

THREE-DIMENSIONAL STRESS CONCENTRATION AROUND A CYLINDRICAL HOLE IN A SEMI-INFINITE ELASTIC BODY

by

Carl K. Youngdahl
Argonne National Laboratory
and
Eli Sternberg
California Institute of Technology

Division of Engineering and Applied Science
CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

May 1965

Three-dimensional stress concentration around a cylindrical hole in a semi-infinite elastic body*

by

Carl K. Youngdahl and Eli Sternberg

Argonne National Laboratory

and

California Institute of Technology

* This investigation was carried out under the auspices of the United States Atomic Energy Commission and was supported in part by the Office of Naval Research under Contract Nonr-220(58).

Summary

This paper contains a three-dimensional solution, exact within classical elastostatics, for the stresses and deformations arising in a halfspace with a semi-infinite transverse cylindrical hole, if the body--at infinite distances from its cylindrical boundary--is subjected to an arbitrary uniform plane field of stress that is parallel to the bounding plane. The solution presented is in integral form and is deduced with the aid of the Papkovich stress functions by means of an especially adapted, unconventional, integral-transform technique. Numerical results for the non-vanishing stresses along the boundary of the hole and for the normal displacement at the plane boundary, corresponding to several values of Poisson's ratio, are also included. These results exhibit in detail the three-dimensional stress boundary layer that emerges near the edges of the hole in the analogous problem for a plate of finite thickness, as the ratio of the plate-thickness to the diameter of the hole grows beyond bounds. The results obtained thus illustrate the limitations inherent in the two-dimensional plane-strain treatment of the spatial plane problem; in addition, they are relevant to failure considerations and are of interest in connection with experimental stress analysis.

Introduction. Motivation of this investigation.

The "plane problem" in the linear equilibrium theory of homogeneous and isotropic elastic solids is in fact a three-dimensional boundary-value problem of formidable complexity. It consists in the determination of the displacements and stresses throughout an elastic body of cylindrical (or prismatic) shape if the surface tractions are confined to the lateral boundary, the terminal cross-sections being free from tractions, provided the prescribed body forces and surface loads are applied at right angles to the generators of the lateral boundary and do not vary in a direction parallel to these generators.[1]

As is well known, the plane-strain and plane-stress solutions associated with the spatial plane problem satisfy the governing differential equations rigorously but meet all of the boundary conditions only in highly exceptional circumstances: in general, they yield merely approximations to the desired three-dimensional solution, which are of different and limited applicability. The plane-strain solution conforms to the lateral boundary conditions, but ordinarily fails to clear the ends of the cylinder from normal tractions. On the other hand, the plane-stress solution fulfills the requirement of traction-free ends but usually violates the lateral boundary conditions, which in this instance are satisfied merely in the thickness-mean. Finally, the associated generalized plane-stress solution furnishes the thickness-averages of the desired displacements and stresses within the ordinarily approximative assumption that the axial normal stress vanishes identically. Further, these averages coincide with the corresponding average displacements and stresses obtainable from the plane-stress solution.

In accordance with the preceding commonplace observations, in general neither the required displacement field nor its accompanying stress field are plane and both fields vary from one cross-section of the body to

---

[1] We are at present concerned exclusively with the second plane boundary-value problem, in which the surface tractions are prescribed over the entire lateral boundary.

another. Exceptions to this statement arise if (a) Poisson's ratio is zero or (b) the axial normal stress predicted by the plane-strain solution is either constant or a linear function of rectangular cartesian coordinates chosen within a representative cross-section. In both of these special cases the plane-stress solution is the exact three-dimensional solution of the original problem. In case (a) the exact solution to the plane problem is also identical with the associated plane-strain solution, whereas in case (b) the stresses found in the plane-strain solution are exact, except possibly for the axial normal stress which here again vanishes identically in the actual solution. A particularly important example of case (b) is supplied by the axisymmetric plane problem when there are no body forces present. In the absence of the degeneracies just described, the generalized plane-stress solution supplies a useful approximation to the desired stresses and deformations if the ratio of the length of the cylinder (thickness of the plate) to the relevant cross-sectional dimension is sufficiently small. In contrast, the plane-strain solution--modified, if necessary, by superposition of the solution corresponding to a uni-axial stress field so as to assure the self-equilibrance of the residual tractions on the ends of the cylinder--yields an approximation appropriate to the central portion of a sufficiently long cylinder.

General methods for dealing with three-dimensional aspects of the plane problem have engaged the attention of several investigators. Thus Reissner [1][2] (1942) proposed a semi-direct variational method aiming at three-dimensional corrections for the theory of generalized plane stress,

[2] Numbers in brackets refer to the list of publications at the end of this paper.

applicable to relatively thin plates. Green [2] (1949), employing infinite series of exact solutions to the elastostatic field equations, developed a formal scheme for coping with a class of spatial boundary-value problems that includes the plane problem. Reiss and Locke [3] (1961), motivated by the objectives of [1] and using the generalized plane-stress solution as a zero-order approximation, pursued the determination of the desired corrections on the basis of a formal expansion of the stress field in powers of the thickness parameter; in this connection they adopted a boundary-layer technique originally devised by Friedrichs [4] (1949) for the analogous plate flexure problem. The possibility of establishing successive approximations to the exact solution of the plane problem with the aid of expansions in powers of Poisson's ratio--using the plane-strain solution as the zero-order approximation--was considered and found impractical by Sternberg and Muki [5] (1959).

As far as the three-dimensional treatment of specific plane problems is concerned, most efforts have been directed at the problem presented by the stress concentration around a transverse circular cylindrical hole in an infinite slab under uniform loads at infinity--a problem of particular engineering interest.[3] The well-known corresponding two-dimensional solutions are due to Kirsch [7] (1898). A three-dimensional solution,

---

[3] The only other plane problem that appears to have been so considered is that of an infinite slab subjected to an internal transverse line load of uniform intensity, for which Green and Willmore [6] (1948) deduced an exact solution in integral form. Although more easily tractable, this problem is less attractive as a vehicle for studying the limitations of the two-dimensional theory since it fails to involve a dimensionless thickness parameter; it is also less interesting from a physical point of view.

which is in infinite series form and whose structure is highly complicated, was deduced by Green [8] (1948) for the non-trivial case of a uni-axial loading at infinity[4] prior to the publication of his general paper [2]. The solution arrived at in [8] satisfies the requisite differential equations rigorously; the accompanying boundary conditions are met to a satisfactory degree of approximation by means of an iterative procedure. A rather crude and quantitatively inadequate three-dimensional analysis of the same problem was carried out independently by Sternberg and Sadowsky [9] (1949), who employed a modification of the Ritz energy method. Alblas [10] (1957) returned to the problem under discussion. Applying the systematic scheme proposed by Green in [2], he succeeded in constructing a series representation of the three-dimensional solution which is more convenient than that contained in [8] and performed extensive numerical evaluations.[5] Finally, Reiss [11] (1963) applied the expansion technique previously developed in [3] to the determination of three-dimensional corrections for the generalized plane-stress solution given in [7].

As is to be anticipated on intuitive grounds, and is confirmed by the numerical results presented in [9], [10], the distribution of the stresses across the plate thickness becomes increasingly more sensitive to changes in the thickness-ratio, i.e. the ratio of the plate thickness to the diameter

---

[4] In the special instance of isotropic tension or compression at infinity, the plane-stress solution constitutes the exact three-dimensional solution. Cf. our earlier remark concerning the rotationally symmetric plane problem.

[5] In [10] Alblas also dealt comprehensively with the analogous flexure problem, whose history is beyond our present scope.

of the hole, at comparatively large values of this ratio. As the thickness-ratio approaches infinity, the transverse variation of the stresses near the cylindrical boundary becomes progressively more pronounced and localized within boundary layers adjacent to the plate faces. Further, the foregoing three-dimensional boundary-layer effect is inadequately represented by the existing solutions to the problem at hand and requires separate treatment for its reliable quantitative appraisal. This leads one to inquire into the stresses and deformations in an elastic half-space with a semi-infinite transverse circular cylindrical hole, due to an arbitrary plane field of stress that is applied parallel to the bounding plane at infinite distances from the axis of the hole.

The solution to the half-space problem just described, which is our main objective, thus supplements the earlier three-dimensional results appropriate to the plate of finite thickness-ratio and supplies further insight into the limitations attached to the conventional two-dimensional treatment of plane problems. In addition, the present problem possesses a twofold intrinsic interest. First, a knowledge of the tri-axiality inherent in the stress distribution near the edges of the hole is desirable from the point of view of failure considerations. Second, the three-dimensional effects sought here, which are absent when Poisson's ratio vanishes, are bound to depend sensitively upon the value of this physical parameter; consequently, the results also illustrate the difficulties that may be encountered in the interpretation of experimental stress-analysis findings when Poisson's ratio of the model used differs appreciably from that of the prototype.

At first sight the half-space problem to be treated presently might appear to be simpler--or, at least, no more complicated--than its counterpart for the plate of finite thickness. This expectation appears to be unwarranted. In the plate problem[6] it is possible, by means of the Papkovich stress functions and separation of variables, to construct a triply infinite aggregate of solutions to the governing differential equations, each member of which clears the plate faces from tractions and corresponds to vanishing loads at infinity. With the aid of appropriate Fourier expansions in the thickness coordinate one may thus reduce the task of removing the residual tractions to which the associated plane-stress solution gives rise on the cylindrical boundary to the solution of a triply infinite system of linear algebraic equations for the unknown coefficients of superposition. Such an expansion scheme is no longer applicable when the range of the thickness coordinate is unbounded; nor is the half-space problem at hand amenable to a treatment by standard integral-transform techniques.

The method of solution adopted in this paper may be outlined briefly as follows. We first use the associated plane-strain solution to transform the original problem into one governed by prescribed normal tractions on the plane boundary and otherwise vanishing loads. To cope with this "residual problem of plane strain", we then employ the Papkovich stress functions to construct a solution in integral form to the elastostatic field equations that clears the plane boundary from shearing tractions, yields vanishing stresses at infinite distances from the hole, and involves four arbitrary weight functions of the integration parameter.

---

[6] See [2], [10].

Further, this solution, which is not obtainable by separation of variables alone, is so constructed as to insure that the application of the remaining four boundary conditions leads to a system of simultaneous integral equations for the unknown weight functions which is reducible to a single one-dimensional integral equation of Fredholm's second kind. The latter reduction is accomplished with the aid of the inversion theorem for the Fourier transform, as well as by recourse to an inversion formula that is closely related to Weber's integral theorem.[7] In this manner we arrive at an integral representation for the exact solution to the original problem in terms of the solution of the foregoing integral equation and otherwise involving only known functions.

The final integral equation was solved numerically on an electronic computer for values of Poisson's ratio of 1/4 and 1/2, the solution of the problem corresponding to zero Poisson's ratio being known beforehand. Unfortunately, the subsequent evaluation of the improper integrals for the desired stresses and displacements, as well as the numerical check on the boundary conditions, did not yield to routine numerical integration methods. To overcome the convergence difficulties encountered, it was essential to remove in closed form certain contributions to the required integrals that impede their convergence. This, in turn, necessitated a detailed examination of the asymptotic behavior of the integrands concerned.

As is suggested by the preceding remarks, the burden of the numerical analysis of the solution established was at least equal to the

---

[7] See Watson [12], p. 468. A similar modification of Weber's theorem was applied by Blenkarn and Wilhoit [13] in connection with a related, axisymmetric, problem for the half-space with a cylindrical bore. In [13] the loading consists of a uniform band of pressure applied at the entrance of the hole.

effort expended on its derivation. There is ample indication to suppose that the numerical results obtained are accurate well beyond the limits of physical relevance.

1. Formulation of problem. Reduction to a residual problem.

Let (x_1, x_2, x_3) be rectangular cartesian coordinates and R the region of space characterized by

$$0 \leq x_3 < \infty , \quad r = (x_1^2 + x_2^2)^{1/2} \geq a , \tag{1.1}$$

so that R is a half-space with a semi-infinite transverse circular cylindrical opening of radius a (Figure 1). Let Π $(x_3 = 0,\ a \leq r < \infty)$ be the plane portion of the boundary of R and Γ $(r = a,\ 0 \leq x_3 < \infty)$ its cylindrical part. Assume further that R is occupied by a homogeneous and isotropic elastic solid with the shear modulus μ and Poisson's ratio ν. With reference to the preceding choice of coordinates, and in the usual indicial notation, the problem to be considered admits the following formulation:

Throughout the interior of R the cartesian components of displacement and stress, u_i and σ_{ij}, must satisfy the displacement equations of equilibrium

$$u_{i,jj} + \frac{1}{1-2\nu} u_{j,ji} = 0 \qquad (1.2)^8$$

together with the stress-displacement relations

$$\sigma_{ij} = \mu\left(\frac{2\nu}{1-2\nu} \delta_{ij} u_{k,k} + u_{i,j} + u_{j,i}\right) , \tag{1.3}$$

in which δ_{ij} is the Kronecker-delta. Since both Π and Γ are to be free from tractions, the boundary conditions take the form

$$\sigma_{3i} = 0 \text{ on } \Pi \tag{1.4}$$

---

[8] The body forces are assumed to vanish identically.

and

$$\sigma_{ij}n_j = 0 \text{ on } \Gamma \ , \tag{1.5}$$

provided n_j denotes the components of the unit outward normal of Γ. Finally, as $r \to \infty$, the stress field is required to approach a uniform plane state of stress that is parallel to Π. Since, without loss in generality, the principal axes of this limiting state of stress may be assumed coincident with the coordinate axes, the preceding loading condition becomes

$$\sigma_{11} \to \sigma_1, \ \sigma_{22} \to \sigma_2, \ \sigma_{12} \to 0, \ \sigma_{3i} \to 0 \text{ as } r \to \infty \ , \tag{1.6}$$

where σ_1 and σ_2 are prescribed constants.

The complete solution of (1.2) admits the well-known representation in terms of the Papkovich-Neuber stress functions, given by

$$u_i = \frac{1}{2\mu}\left[(\Phi + x_j\Psi_j)_{,i} - 4(1-\nu)\Psi_i\right] \tag{1.7}$$

with

$$\nabla^2\Phi = 0, \ \nabla^2\Psi_i = 0 \ . \tag{1.8}$$

Further, from (1.3), (1.7), (1.8) follows

$$\sigma_{ij} = \Phi_{,ij} - (1-2\nu)(\Psi_{i,j} + \Psi_{j,i}) + x_k\Psi_{k,ij} - 2\nu\delta_{ij}\Psi_{k,k} \ . \tag{1.9}$$

Accordingly, the problem under consideration reduces to the determination of functions Φ and Ψ_i that are harmonic in the interior of R and such that the stresses (1.9) obey (1.4), (1.5), (1.6).

Let S, with the cartesian components u_i and σ_{ij}, be the desired solution to the foregoing boundary-value problem. Further, let S' and S'' be the respective solutions of (1.2) to (1.6) appropriate to the following two basic loading cases.

Case 1: $\sigma_{11} \to \sigma$, $\sigma_{22} \to \sigma$, $\sigma_{12} \to 0$, $\sigma_{3i} \to 0$ as $r \to \infty$. (1.10)

Case 2: $\sigma_{11} \to \sigma$, $\sigma_{22} \to -\sigma$, $\sigma_{12} \to 0$, $\sigma_{3i} \to 0$ as $r \to \infty$. (1.11)

Thus S' and S'' correspond respectively to a plane isotropic state of stress and to a plane state of pure shear at infinity. Then, clearly,

$$S = \frac{\sigma_1 + \sigma_2}{2\sigma} S' + \frac{\sigma_1 - \sigma_2}{2\sigma} S'' . \qquad (1.12)^9$$

The solution S' to the axisymmetric plane problem arising in Case 1 is elementary, as is the plane-strain solution associated with Case 2. Both of these familiar solutions will be recalled shortly.

We now introduce cylindrical coordinates (r, θ, z) through the mapping

$$\left.\begin{array}{l} x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = z, \\ 0 \le r < \infty,\ 0 \le \theta < 2\pi,\ -\infty < z < \infty, \end{array}\right\} \qquad (1.13)$$

and at the same time define the dimensionless coordinates

$$\rho = r/a\ ,\ \zeta = z/a\ , \qquad (1.14)$$

the dimensionless stress functions

$$\hat{\Phi} = \Phi/a^2\sigma\ ,\ \hat{\Psi}_i = \Psi_i/a\sigma\ , \qquad (1.15)$$

as well as the dimensionless cylindrical components of displacement and stress

---

[9] Here $\sigma \neq 0$ is implied; we avoid the normalization $\sigma = 1$ in order not to obscure the dimensionality of S' and S''. Addition and multiplication by a scalar constant of solutions to the elastostatic field equations are to be interpreted in the sense of the corresponding operations applied to their fields of displacement and stress.

$$\left.\begin{aligned} v_r &= 2\mu u_r/a\sigma,\ v_\theta = 2\mu u_\theta/a\sigma,\ v_z = 2\mu u_z/a\sigma\ , \\ \tau_{rr} &= \sigma_{rr}/\sigma,\ \tau_{\theta\theta} = \sigma_{\theta\theta}/\sigma,\ \tau_{zz} = \sigma_{zz}/\sigma\ , \\ \tau_{\theta z} &= \sigma_{\theta z}/\sigma,\ \tau_{zr} = \sigma_{zr}/\sigma,\ \tau_{r\theta} = \sigma_{r\theta}/\sigma\ . \end{aligned}\right\} \tag{1.16}$$

With this notation, on setting

$$\alpha = 2(1-\nu),\ \hat{\omega} = \hat{\Phi} + \rho\hat{\Psi}_1\cos\theta + \rho\hat{\Psi}_2\sin\theta + \zeta\hat{\Psi}_3\ , \tag{1.17}$$

equations (1.7) in cylindrical coordinates[10] assume the form

$$\left.\begin{aligned} v_r &= \frac{\partial\hat{\omega}}{\partial\rho} - 2\alpha(\hat{\Psi}_1\cos\theta+\hat{\Psi}_2\sin\theta)\ , \\ v_\theta &= \frac{1}{\rho}\frac{\partial\hat{\omega}}{\partial\theta} + 2\alpha(\hat{\Psi}_1\sin\theta - \hat{\Psi}_2\cos\theta)\ , \\ v_z &= \frac{\partial\hat{\omega}}{\partial\zeta} - 2\alpha\hat{\Psi}_3\ . \end{aligned}\right\} \tag{1.18}$$

Similarly, (1.9) give way to

---

[10] See [14] for the Papkovich-Neuber solution in general orthogonal curvilinear coordinates.

$$\left.\begin{aligned}
\tau_{rr} &= \frac{\partial^2\hat{\omega}}{\partial\rho^2} + (\alpha-2)\left[-\frac{1}{\rho}\frac{\partial\hat{\Psi}_1}{\partial\theta}\sin\theta + \frac{1}{\rho}\frac{\partial\hat{\Psi}_2}{\partial\theta}\cos\theta + \frac{\partial\hat{\Psi}_3}{\partial\zeta}\right] \\
&\quad - (\alpha+2)\left[\frac{\partial\hat{\Psi}_1}{\partial\rho}\cos\theta + \frac{\partial\hat{\Psi}_2}{\partial\rho}\sin\theta\right], \\
\tau_{\theta\theta} &= \frac{1}{\rho^2}\frac{\partial^2\hat{\omega}}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial\hat{\omega}}{\partial\rho} + (\alpha-2)\left[\frac{\partial\hat{\Psi}_1}{\partial\rho}\cos\theta + \frac{\partial\hat{\Psi}_2}{\partial\rho}\sin\theta + \frac{\partial\hat{\Psi}_3}{\partial\zeta}\right] \\
&\quad + (\alpha+2)\left[\frac{1}{\rho}\frac{\partial\hat{\Psi}_1}{\partial\theta}\sin\theta - \frac{1}{\rho}\frac{\partial\hat{\Psi}_2}{\partial\theta}\cos\theta\right], \\
\tau_{zz} &= \frac{\partial^2\hat{\omega}}{\partial\zeta^2} + (\alpha-2)\left[\frac{\partial\hat{\Psi}_1}{\partial\rho}\cos\theta + \frac{\partial\hat{\Psi}_2}{\partial\rho}\sin\theta - \frac{1}{\rho}\frac{\partial\hat{\Psi}_1}{\partial\theta}\sin\theta\right. \\
&\quad \left. + \frac{1}{\rho}\frac{\partial\hat{\Psi}_2}{\partial\theta}\cos\theta\right] - (\alpha+2)\frac{\partial\hat{\Psi}_3}{\partial\zeta}, \\
\tau_{\theta z} &= \frac{1}{\rho}\frac{\partial^2\hat{\omega}}{\partial\theta\partial\zeta} + \alpha\left[\frac{\partial\hat{\Psi}_1}{\partial\zeta}\sin\theta - \frac{\partial\hat{\Psi}_2}{\partial\zeta}\cos\theta - \frac{1}{\rho}\frac{\partial\hat{\Psi}_3}{\partial\theta}\right], \\
\tau_{zr} &= \frac{\partial^2\hat{\omega}}{\partial\rho\partial\zeta} - \alpha\left[\frac{\partial\hat{\Psi}_1}{\partial\zeta}\cos\theta + \frac{\partial\hat{\Psi}_2}{\partial\zeta}\sin\theta + \frac{\partial\hat{\Psi}_3}{\partial\rho}\right], \\
\tau_{r\theta} &= \frac{1}{\rho}\frac{\partial^2\hat{\omega}}{\partial\rho\partial\theta} - \frac{1}{\rho^2}\frac{\partial\hat{\omega}}{\partial\theta} - \alpha\left[\frac{1}{\rho}\frac{\partial\hat{\Psi}_1}{\partial\theta}\cos\theta + \frac{1}{\rho}\frac{\partial\hat{\Psi}_2}{\partial\theta}\sin\theta\right. \\
&\quad \left. - \frac{\partial\hat{\Psi}_1}{\partial\rho}\sin\theta + \frac{\partial\hat{\Psi}_2}{\partial\rho}\cos\theta\right].
\end{aligned}\right\} \quad (1.19)$$

In connection with (1.8) we cite the cylindrical form of the Laplacian operator

$$a^2\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\zeta^2} . \quad (1.20)$$

The plane-stress solution for Case 1, which coincides with the exact solution S', and the plane-strain solution associated with Case 2, which will hereafter be designated by $\overset{\circ}{S}''$, may be generated with the aid of the Papkovich-Neuber stress functions.[11] As is readily confirmed by means of (1.14) to (1.20), the results may be summarized as follows.

<u>Solution</u> S' (<u>Plane-stress solution for Case 1</u>):

$$\hat{\Phi} = \frac{1-\nu}{1+\nu}\left(\frac{\rho^2}{2} - \zeta^2\right) + \log\rho, \quad \hat{\Psi}_1 = \hat{\Psi}_2 = 0, \quad \hat{\Psi}_3 = -\frac{1}{1+\nu}\zeta, \tag{1.21}$$

$$\left.\begin{aligned} &v_r = \frac{1-\nu}{1+\nu}\rho + \frac{1}{\rho}, \quad v_\theta = 0, \quad v_z = -\frac{2\nu}{1+\nu}\zeta, \\ &\tau_{rr} = 1 - \frac{1}{\rho^2}, \quad \tau_{\theta\theta} = 1 + \frac{1}{\rho^2}, \quad \tau_{zz} = \tau_{\theta z} = \tau_{zr} = \tau_{r\theta} = 0. \end{aligned}\right\} \tag{1.22}$$

<u>Solution</u> $\overset{\circ}{S}''$ (<u>Plane-strain solution for Case 2</u>):

$$\hat{\Phi} = \frac{1}{2}\left(\rho^2 + \frac{1}{\rho^2}\right)\cos 2\theta, \quad \hat{\Psi}_1 = -\frac{1}{\rho}\cos\theta, \quad \hat{\Psi}_2 = \frac{1}{\rho}\sin\theta, \quad \hat{\Psi}_3 = 0, \tag{1.23}$$

$$\left.\begin{aligned} v_r &= \left[\rho + \frac{4(1-\nu)}{\rho} - \frac{1}{\rho^3}\right]\cos 2\theta, \\ v_\theta &= -\left[\rho + \frac{2(1-2\nu)}{\rho} + \frac{1}{\rho^3}\right]\sin 2\theta, \quad v_z = 0, \\ \tau_{rr} &= \left(1 - \frac{4}{\rho^2} + \frac{3}{\rho^4}\right)\cos 2\theta, \quad \tau_{\theta\theta} = -\left(1 + \frac{3}{\rho^4}\right)\cos 2\theta, \\ \tau_{zz} &= -\frac{4\nu}{\rho^2}\cos 2\theta, \quad \tau_{\theta z} = \tau_{zr} = 0, \quad \tau_{r\theta} = -\left(1 + \frac{2}{\rho^2} - \frac{3}{\rho^4}\right)\sin 2\theta. \end{aligned}\right\} \tag{1.24}$$

The plane-strain solution $\overset{\circ}{S}''$ satisfies the elastostatic field equations, conforms to the loading condition (1.11) for Case 2, and meets all of the boundary conditions (1.4), (1.5) with the exception of the requirement

---

[11] See, for example, Timoshenko-Goodier [15], for a derivation of these classical solutions on the basis of the Airy stress function.

$\sigma_{33} = 0$ on Π. This requirement is fulfilled by $\mathring{S}''$ if and only if $\nu = 0$, in which instance $\mathring{S}''$ is identical with S''. We now set

$$S'' = \mathring{S}'' + \overline{S} \tag{1.25}$$

and call $\overline{S}$, so defined, the solution to the "residual problem" of plane strain for Case 2. It is clear from (1.25) and the remarks preceding (1.25) that this residual problem is governed by the boundary conditions

$$\tau_{rr} = \tau_{r\theta} = \tau_{rz} = 0 \text{ on } \rho = 1 \quad (0 \leq \zeta < \infty), \tag{1.26}$$

$$\tau_{zr} = \tau_{z\theta} = 0, \; \tau_{zz} = \frac{2(2-\alpha)}{\rho^2}\cos 2\theta \text{ on } \zeta = 0 \quad (1 \leq \rho < \infty), \tag{1.27}$$

and the regularity conditions

$$\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}, \tau_{\theta z}, \tau_{zr}, \tau_{r\theta} \to 0 \text{ as } \rho \to \infty \quad (0 \leq \zeta < \infty), \tag{1.28}$$

all of which must hold identically for $0 \leq \theta < 2\pi$.

Bearing in mind (1.12) and (1.25), we note that the determination of the solution S to the original problem has been reduced to the task of constructing stress functions $\hat{\Phi}$ and $\hat{\Psi}_i$, harmonic in the interior of R, that generate--in the sense of (1.19)--a stress distribution satisfying (1.26), (1.27), (1.28).

2. Reduction of residual problem to a one-dimensional integral equation.

Our next objective is to construct stress functions suited for the solution of the residual problem. To this end, guided by the θ-dependence of $\tau_{zz}(\rho, \theta, 0)$ in (1.27) and by the manner in which θ enters (1.19), we set

$$\left.\begin{aligned} \hat{\Phi}(\rho,\theta,\zeta) &= \varphi(\rho,\zeta)\cos 2\theta, \\ \hat{\Psi}_1(\rho,\theta,\zeta) &= \chi(\rho,\zeta)\cos\theta, \\ \hat{\Psi}_2(\rho,\theta,\zeta) &= -\chi(\rho,\zeta)\sin\theta, \\ \hat{\Psi}_3(\rho,\theta,\zeta) &= \psi(\rho,\zeta)\cos 2\theta. \end{aligned}\right\} \tag{2.1}$$

On defining an auxiliary function ω through

$$\omega = \varphi + \rho\chi + \zeta\psi \ , \tag{2.2}$$

we infer from (2.1) and (1.17), (1.18), (1.19) that now

$$\left.\begin{aligned} v_r &= \left[\frac{\partial\omega}{\partial\rho} - 2\alpha\chi\right]\cos 2\theta, \quad v_\theta = \left[-\frac{2\omega}{\rho} + 2\alpha\chi\right]\sin 2\theta, \\ v_z &= \left[\frac{\partial\omega}{\partial\zeta} - 2\alpha\psi\right]\cos 2\theta, \end{aligned}\right\} \tag{2.3}$$

$$\left.\begin{aligned} \tau_{rr} &= \left[\frac{\partial^2\omega}{\partial\rho^2} + (\alpha-2)\left(\frac{\partial\psi}{\partial\zeta} - \frac{\chi}{\rho}\right) - (\alpha+2)\frac{\partial\chi}{\partial\rho}\right]\cos 2\theta, \\ \tau_{\theta\theta} &= \left[\frac{1}{\rho}\frac{\partial\omega}{\partial\rho} - \frac{4\omega}{\rho^2} + (\alpha-2)\left(\frac{\partial\psi}{\partial\zeta} + \frac{\partial\chi}{\partial\rho}\right) + (\alpha+2)\frac{\chi}{\rho}\right]\cos 2\theta, \\ \tau_{zz} &= \left[\frac{\partial^2\omega}{\partial\zeta^2} + (\alpha-2)\left(\frac{\partial\chi}{\partial\rho} - \frac{\chi}{\rho}\right) - (\alpha+2)\frac{\partial\psi}{\partial\zeta}\right]\cos 2\theta, \\ \tau_{\theta z} &= \left[-\frac{2}{\rho}\frac{\partial\omega}{\partial\zeta} + \alpha\frac{\partial\chi}{\partial\zeta} + 2\alpha\frac{\psi}{\rho}\right]\sin 2\theta, \\ \tau_{zr} &= \left[\frac{\partial^2\omega}{\partial\rho\partial\zeta} - \alpha\frac{\partial\chi}{\partial\zeta} - \alpha\frac{\partial\psi}{\partial\rho}\right]\cos 2\theta, \\ \tau_{r\theta} &= \left[-2\frac{\partial}{\partial\rho}\left(\frac{\omega}{\rho}\right) + \alpha\left(\frac{\partial\chi}{\partial\rho} + \frac{\chi}{\rho}\right)\right]\sin 2\theta \ . \end{aligned}\right\} \tag{2.4}$$

Further, in view of (1.20), the stress functions Φ, Ψ_i determined by (2.1), (1.15) are harmonic in the interior of R if and only if the functions φ, χ, ψ on $(1,\infty) \times (0,\infty)$ satisfy

$$\nabla_1^2\chi = 0, \ \nabla_2^2\varphi = \nabla_2^2\psi = 0, \tag{2.5}$$

where

$$\nabla_n^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\zeta^2} - \frac{n^2}{\rho^2} \qquad (n = 1, 2) \ . \tag{2.6}$$

Separation of variables applied to the equation

$$\nabla_n^2 F = 0 \qquad (n = 1, 2) \tag{2.7}$$

at once yields the real-valued product solutions

$$\left.\begin{aligned} F(\rho, \zeta) &= [J_n(\gamma\rho) \text{ or } Y_n(\gamma\rho)][\exp(\gamma\zeta) \text{ or } \exp(-\gamma\zeta)], \\ F(\rho, \zeta) &= [I_n(\gamma\rho) \text{ or } K_n(\gamma\rho)][\cos(\gamma\zeta) \text{ or } \sin(\gamma\zeta)], \end{aligned}\right\} \tag{2.8}$$

in which J_n, Y_n and I_n, K_n are, respectively, the ordinary and the modified Bessel functions of the first and second kind, of order n, while γ is an arbitrary non-negative constant. For future reference we also note that if F satisfies (2.7), the same is true of

$$F^* = \rho\frac{\partial F}{\partial \rho} + \zeta\frac{\partial F}{\partial \zeta} \; . \qquad (2.9)^{12}$$

This observation enables one to deduce from (2.8) solutions of (2.7) that are not obtainable by separation of variables.

We now seek to determine functions φ, χ, ψ that satisfy (2.5) and generate--in the sense of (2.4)--a stress field meeting the first two of the boundary conditions (1.27), i.e. giving rise to vanishing shearing tractions on $\zeta = 0$; in addition, the resulting stress field must tend to zero as $\rho \to \infty$, in accordance with (1.28), and is to remain bounded as $\zeta \to \infty$. With this purpose in mind we note first from (2.2), (2.4) that τ_{zr}, $\tau_{z\theta}$ are both identically zero at $\zeta = 0$ if φ, χ are even functions of ζ, while ψ is odd in ζ, and also if

$$\frac{\partial \varphi}{\partial \zeta} = (\alpha - 1)\psi \; , \; \chi = 0 \; . \tag{2.10}$$

In view of (2.8), and because of the regularity requirements[13] mentioned

---

[12] Recall that the harmonicity of a function H implies that of $\underline{x}\cdot\nabla H$, if $\underline{x}$ is the position vector.

[13] Recall that $I_n(\gamma\rho)$ becomes unbounded as $\rho \to \infty$.

before, we are thus led to set

$$\left.\begin{aligned} \varphi &= \varphi_1 + (\alpha-1)\varphi_2 , \\ \chi &= \chi_1 , \\ \psi &= \psi_1 + \frac{\partial\varphi_2}{\partial\zeta} , \end{aligned}\right\} \tag{2.11}$$

with

$$\left.\begin{aligned} \varphi_1(\rho,\zeta) &= \int_0^\infty A^*(\gamma)\, K_2(\gamma\rho) \cos(\gamma\zeta) d\gamma , \\ \chi_1(\rho,\zeta) &= \int_0^\infty B^*(\gamma)\, K_1(\gamma\rho) \cos(\gamma\zeta) d\gamma , \\ \psi_1(\rho,\zeta) &= \int_0^\infty C^*(\gamma)\, K_2(\gamma\rho) \sin(\gamma\zeta) d\gamma , \end{aligned}\right\} \tag{2.12}$$

in which A^*, B^*, C^* are as yet arbitrary weight-functions of the integration parameter γ.

It is apparent from (2.12), (2.2), (2.4) that the contributions of φ_1, χ_1, ψ_1 to all but the leading term in $\tau_{rr}(1,\theta,\zeta)$, $\tau_{r\theta}(1,\theta,\zeta)$, and $\tau_{rz}(1,\theta,\zeta)$ take the form of Fourier integrals, whose respective integrands involve ζ only through the factor $\cos(\gamma\zeta)$ or $\sin(\gamma\zeta)$. In contrast, the terms contributed through ω fail to exhibit such a structure. To remove this deficiency, which would prevent us from taking advantage of the Fourier-transform inversion theorem in coping with the boundary conditions (1.26), we replace the definition of φ_1 in (2.12) by

$$\varphi_1(\rho,\zeta) = \int_0^\infty \{A^*(\gamma)K_2(\gamma\rho)\cos(\gamma\zeta)$$

$$+ C^*(\gamma)[\rho K_2'(\gamma\rho)\cos(\gamma\rho) - \zeta K_2(\gamma\rho)\sin(\gamma\zeta)]\}d\gamma\ . \qquad (2.13)^{14}$$

The integrand in (2. 13) is an even function of ζ that possesses the required regularity as $\rho \to \infty$ and satisfies (2. 5), as is clear from (2. 8), (2. 9). Further, with this choice of φ_1, and with χ_1, ψ_1 defined as in (2. 12), one now has

$$\omega_1(\rho,\zeta) \equiv \varphi_1(\rho,\zeta)+\rho\chi_1(\rho,\zeta)+\zeta\psi_1(\rho,\zeta)$$

$$= \int_0^\infty [A^*(\gamma)K_2(\gamma\rho)+B^*(\gamma)\rho K_1(\gamma\rho)+C^*(\gamma)\rho K_2'(\gamma\rho)]\cos(\gamma\zeta)d\zeta\ , \qquad (2.14)$$

so that our immediate task has been accomplished.

We have yet to dispose of φ_2 in (2. 11). The following choice of φ_2 is admissible by virtue of the first of (2. 8); it is motivated by the form of the boundary condition on τ_{zz} in (1. 27), as will become clear later on. We take

$$\varphi_2(\rho,\zeta) = \int_0^\infty D^*(\gamma)\Omega_2(\gamma,\rho)\exp(-\gamma\zeta)d\gamma\ , \qquad (2.15)$$

where

$$\Omega_2(\gamma,\rho) = Y_2'(\gamma)J_2(\gamma\rho) - J_2'(\gamma)Y_2(\gamma\rho)\ , \qquad (2.16)$$

while D^* is yet another initially arbitrary function of the integration parameter. To shorten future results it is convenient to define a new quadruplet of weight-functions A, B, C, D through

$$A^* = A - \frac{2}{\alpha}C,\ B^* = -\frac{\gamma}{\alpha}C,\ C^* = \gamma B - \frac{\gamma}{\alpha}C,\ D^* = -\gamma^2 D\ . \qquad (2.17)$$

---

[14] Throughout this paper K_n', J_n', and Y_n' denote the first derivatives of the corresponding Bessel functions.

In view of (2.11), the last two of (2.12), and (2.13), (2.15), (2.17), as well as the recurrence relation expressing K_1 in terms of K_2 and K_2', we arrive at the final choice of φ, χ, ψ:

$$\left.\begin{aligned}
\varphi(\rho,\zeta) &= \int_0^\infty \{[A(\gamma) - \tfrac{2}{\alpha}C(\gamma)]K_2(\gamma\rho)\cos(\gamma\zeta) \\
&\quad + [B(\gamma) - \tfrac{1}{\alpha}C(\gamma)][\gamma\rho K_2'(\gamma\rho)\cos(\gamma\zeta) - \gamma\zeta K_2(\gamma\rho)\sin(\gamma\zeta)] \\
&\quad + (1-\alpha)\gamma^2 D(\gamma)\Omega_2(\gamma\rho)\exp(-\gamma\zeta)\}d\gamma\ , \\
\chi(\rho,\zeta) &= \frac{1}{\alpha}\int_0^\infty C(\gamma)[\tfrac{2}{\rho}K_2(\gamma\rho) + \gamma K_2'(\gamma\rho)]\cos(\gamma\zeta)d\gamma\ , \\
\psi(\rho,\zeta) &= \int_0^\infty \{[B(\gamma) - \tfrac{1}{\alpha}C(\gamma)]\gamma K_2(\gamma\rho)\sin(\gamma\zeta) \\
&\quad + \gamma^3 D(\gamma)\Omega_2(\gamma,\rho)\exp(-\gamma\zeta)\}d\gamma\ .
\end{aligned}\right\} \qquad (2.18)$$

The four weight-functions A, B, C, D in (2.18) are to be determined from the remaining four boundary conditions, i.e. in accordance with (1.26) and the last of (2.4). We therefore substitute formally from (2.18) into the first and the last two of (2.4). In this manner, using the recurrence relations for K_n, the modified Bessel equation, as well as the identity

$$\Omega_2(\gamma,1) = \frac{2}{\pi\gamma}\ , \qquad (2.19)[15]$$

and adopting the auxiliary notation

$$K(\gamma) = \gamma\,\frac{K_2'(\gamma)}{K_2(\gamma)}\ , \qquad (2.20)$$

---

[15] See [16], p. 79, No. 28.

we find that $\tau_{rr}(1,\theta,\zeta)$, $\tau_{r\theta}(1,\theta,\zeta)$, $\tau_{rz}(1,\theta,\zeta)$ vanish identically provided, for $0 \le \zeta < \infty$,

$$\int_0^\infty \{A(\gamma)[\gamma^2+4-K(\gamma)]+B(\gamma)[(\alpha-1)\gamma^2-4+(\gamma^2+4)K(\gamma)]$$

$$-2C(\gamma)[\gamma^2+2+K(\gamma)]\}K_2(\gamma)\cos(\gamma\zeta)d\gamma$$

$$=\frac{2}{\pi}\int_0^\infty \gamma D(\gamma)[4(\alpha-1-\gamma\zeta)+\gamma^2(\gamma\zeta-1)]\exp(-\gamma\zeta)d\gamma \ , \tag{2.21}$$

$$\int_0^\infty \{2A(\gamma)[1-K(\gamma)]+2B(\gamma)[-(\gamma^2+4)+K(\gamma)]$$

$$+C(\gamma)[\gamma^2+4+2K(\gamma)]\}K_2(\gamma)\cos(\gamma\zeta)d\gamma$$

$$=\frac{4}{\pi}\int_0^\infty \gamma D(\gamma)(\alpha-1-\gamma\zeta)\exp(-\gamma\zeta)d\gamma \ , \tag{2.22}$$

$$\int_0^\infty \{A(\gamma)K(\gamma)+B(\gamma)[\gamma^2+4+\alpha K(\gamma)]$$

$$-2C(\gamma)[1+K(\gamma)]\}\gamma K_2(\gamma)\sin(\gamma\zeta)d\gamma = 0 \ . \tag{2.23}$$

Similarly $\tau_{zz}(\rho,\theta,0)$ is found to vanish identically if, for $1 \le \rho < \infty$,

$$\int_0^\infty \gamma^4 D(\gamma)\Omega_2(\gamma,\rho)d\gamma = \frac{2(2-\alpha)}{\rho^2}+\int_0^\infty \{[A(\gamma)+(\alpha+2)B(\gamma)-2C(\gamma)]K_2(\gamma\rho)$$

$$+B(\gamma)\gamma\rho K_2'(\gamma\rho)\}\gamma^2 d\gamma \ . \tag{2.24}$$

Equations (2.21) to (2.24) constitute a system of four simultaneous linear integral equations for the unknown functions A, B, C, D. With a view toward reducing this system to a single integral equation, we recall first that the inversion theorem for the Fourier cosine and sine transforms[16]

---

[16] See, for example, [17], p. 17.

furnishes the identity

$$f(\eta) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(\gamma)\begin{Bmatrix}\cos(\gamma\zeta)\cos(\eta\zeta)\\ \sin(\gamma\zeta)\sin(\eta\zeta)\end{Bmatrix} d\gamma d\zeta \qquad (0<\eta<\infty), \tag{2.25}$$

valid for every f which is suitably well-behaved on $[0,\infty)$. Further, we note the particular Fourier cosine transforms[17]

$$\left.\begin{aligned} &\int_0^\infty \gamma \exp(-\gamma\zeta)\cos(\eta\zeta)d\zeta = g(\eta/\gamma)\ , \\ &\int_0^\infty \gamma(1+\gamma\zeta)\exp(-\gamma\zeta)\cos(\eta\zeta)d\zeta = 2g^2(\eta/\gamma)\ , \end{aligned}\right\} \tag{2.26}$$

where

$$g(x) = (1+x^2)^{-1}\ . \tag{2.27}$$

We now multiply (2.21), (2.22) by $(2/\pi)\cos(\eta\zeta)$, (2.23) by $(2/\pi)\sin(\eta\zeta)$, integrate the resulting equations with respect to ζ over the range $[0,\infty)$, and use (2.25), (2.26), (2.27) to obtain

$$\left.\begin{aligned} &A(\eta)[\eta^2+4-K(\eta)] + B(\eta)[(\alpha-1)\eta^2-4+(\eta^2+4)K(\eta)] \\ &-2C(\eta)[\eta^2+2+K(\eta)] = \frac{8}{\pi^2 K_2(\eta)}\int_0^\infty D(\gamma)[2\alpha-(\eta^2+4)g(\eta/\gamma)]g(\eta/\gamma)d\gamma\ , \\ &2A(\eta)[1-K(\eta)] + 2B(\eta)[-(\eta^2+4)+K(\eta)] \\ &+C(\eta)[\eta^2+4+2K(\eta)] = \frac{8}{\pi^2 K_2(\eta)}\int_0^\infty D(\gamma)[\alpha-2g(\eta/\gamma)]g(\eta/\gamma)d\gamma\ , \\ &A(\eta)K(\eta)+B(\eta)[\eta^2+4+\alpha K(\eta)]-2C(\eta)[1+K(\eta)] = 0\ . \end{aligned}\right\} \tag{2.28}$$

---

[17] See [18], p. 14.

Equations (2.28) are three simultaneous linear algebraic equations in A, B, and C; their solution, after a convenient change of the dummy variables, is given by

$$\left.\begin{aligned}
A(\gamma) &= \frac{16\alpha}{\pi^2 K_2(\gamma)\Delta(\gamma)}[f_1(\gamma)+f_4(\gamma)]D_1(\gamma) \\
&- \frac{8}{\pi^2 K_2(\gamma)\Delta(\gamma)}\{[\gamma^2+4+\alpha K(\gamma)]f_3(\gamma)-4\alpha f_2(\gamma)[K(\gamma)+1]\}D_2(\gamma)\ , \\
B(\gamma) &= \frac{16\alpha}{\pi^2 K_2(\gamma)\Delta(\gamma)} f_2(\gamma)D_1(\gamma)+\frac{8}{\pi^2 K_2(\gamma)\Delta(\gamma)} K(\gamma)f_3(\gamma)D_2(\gamma)\ , \\
C(\gamma) &= \frac{8\alpha}{\pi^2 K_2(\gamma)\Delta(\gamma)} f_1(\gamma)D_1(\gamma)+\frac{16\alpha}{\pi^2 K_2(\gamma)\Delta(\gamma)} K(\gamma)f_2(\gamma)D_2(\gamma)\ ,
\end{aligned}\right\} \quad (2.29)$$

where

$$D_1(\gamma) = \int_0^\infty D(\xi)g(\gamma/\xi)d\xi\ ,\quad D_2(\gamma) = \int_0^\infty D(\xi)g^2(\gamma/\xi)d\xi\ , \qquad (2.30)$$

$$\left.\begin{aligned}
f_1(\gamma) &= \gamma^2[\gamma^2+4-K^2(\gamma)]+3\alpha K^2(\gamma)\ , \\
f_2(\gamma) &= \gamma^2-(\gamma^2+3)K(\gamma)\ , \\
f_3(\gamma) &= (\gamma^2+2)(\gamma^2+6)-2\gamma^2K(\gamma)\ , \\
f_4(\gamma) &= (\gamma^2+3)(\gamma^2+4)-\gamma^2K(\gamma)-\alpha f_2(\gamma)+3\alpha K(\gamma)\ , \\
\Delta(\gamma) &= [\gamma^2+4-K^2(\gamma)]f_3(\gamma)+\alpha[-4\gamma^2+8(\gamma^2+3)\ K(\gamma)-\gamma^2K^2(\gamma)-6K^3(\gamma)]\ .
\end{aligned}\right\} \quad (2.31)$$

Turning to (2.24), we cite the following inversion identity, which is a modified form of Weber's integral theorem[18] and holds true for any function f sufficiently regular on $[0,\infty)$ and every $n\ (n=0,1,2,\ldots)$:

---

[18] See Watson [12], p. 468 and Titchmarsh [19].

$$f(\eta)\{[J_n'(\eta)]^2+[Y_n'(\eta)]^2\} = \int_1^\infty \int_0^\infty \gamma\rho f(\gamma)\Omega_n(\gamma,\rho)\Omega_n(\eta,\rho)d\gamma d\rho \ , \tag{2.32}$$

where

$$\Omega_n(\gamma,\rho) = Y_n'(\gamma)J_n(\gamma\rho)-J_n'(\gamma)Y_n(\gamma\rho) \ . \tag{2.33}$$

A proof of (2.32), restricted to $n = 0$, was given by Blenkarn [20][19], who used the inversion formula (2.32) with this value of n in connection with a rotationally symmetric half-space problem described in the Introduction. A proof applicable to arbitrary integral values of n is contained in the Appendix of the present paper. For the problem we are considering the relevant value of n is two in view of (2.24), $\Omega_2(\gamma,\rho)\exp(-\gamma\zeta)$ being a solution of (2.7) with $n = 2$. Indeed, as is now evident, the choice of φ_2 in (2.15) was motivated by the fact that (2.32) enables one to reduce (2.24)--with A, B, C given by (2.29)--to a Fredholm equation for the unknown weight-function D.

To effect the reduction just alluded to, we shall need to make use of the definite integrals

$$\left.\begin{aligned}
&\int_1^\infty \frac{1}{\rho}\,\Omega_2(\eta,\rho)d\rho = \frac{4}{\pi\eta^3} \ , \\
&\int_1^\infty \rho\Omega_2(\eta,\rho)K_2(\gamma\rho)d\rho = -\frac{2\gamma}{\pi\eta^3}K_2'(\gamma)g(\gamma/\eta) \ , \\
&\int_1^\infty \gamma\rho^2\Omega_2(\eta,\rho)K_2'(\gamma\rho)d\rho = \frac{2}{\pi\eta^3}\{-(\gamma^2+4)K_2(\gamma) \\
&\qquad\qquad + 2\gamma K_2'(\gamma)[1-g(\gamma/\eta)]\}g(\gamma/\eta) \ ,
\end{aligned}\right\} \tag{2.34}$$

in which g is once again given by (2.27). Equations (2.34) may be deduced

---

[19] For a condensed published version of [20], see Blenkarn and Wilhoit [13].

from the indefinite integral

$$\int \rho\Omega_2(\eta,\rho)K_2(\gamma\rho)d\rho =$$

$$\frac{\rho}{\gamma^2+\eta^2}[\Omega_2(\eta,\rho)\frac{\partial}{\partial\rho}K_2(\gamma\rho)-K_2(\gamma\rho)\frac{\partial}{\partial\rho}\Omega_2(\eta,\rho)] , \qquad (2.35)$$

which can be verified directly by differentiation and an appeal to the ordinary and the modified Bessel equation. Thus, the second of (2.34) is obtained by evaluating (2.35) for the appropriate limits of integration, taking account of (2.33) and (2.19); the first of (2.34) follows from the second if one multiplies the latter by γ^2 and then passes to the limit as $\gamma \to 0$; finally, differentiation of the second of (2.34) with respect to γ, in conjunction with the modified Bessel equation, yields the third.

Now multiply (2.24) by $\rho\Omega_2(\eta,\rho)$, integrate the resulting identity with respect to ρ over the range $[1,\infty)$ and invoke (2.32), (2.34) to confirm that

$$D(\eta)\eta^3\{[J_2'(\eta)]^2+[Y_2'(\eta)]^2\} =$$

$$\frac{8(2-\alpha)}{\pi\eta^3} - \frac{2}{\pi\eta^3}\int_0^\infty \{A(\gamma)K(\gamma)+B(\gamma)[\gamma^2+4+\alpha K(\gamma)] -$$

$$2C(\gamma)K(\gamma)+2B(\gamma)K(\gamma)g(\gamma/\eta)\}g(\gamma/\eta)\gamma^2K_2(\gamma)d\gamma . \qquad (2.36)$$

Substituting for A, B, C from (2.29) into (2.36) and letting

$$Z(\eta) = [J_2'(\eta)]^2+[Y_2'(\eta)]^2 , \qquad (2.37)$$

we arrive at the integral equation

$$D(\eta)\eta^6 Z(\eta) = \frac{8(2-\alpha)}{\pi} + \int_0^\infty D(\xi)L(\xi,\eta)d\xi , \qquad (2.38)$$

whose symmetric kernel L is defined on $[0,\infty) \times [0,\infty)$ through

$$L(\xi,\eta) = -\frac{32}{\pi^3}\int_0^\infty \frac{\gamma^2}{\Delta(\gamma)}\, g(\gamma/\xi)g(\gamma/\eta)\{\alpha f_1(\gamma)$$

$$+2\alpha[g(\gamma/\xi)+g(\gamma/\eta)]K(\gamma)f_2(\gamma)+g(\gamma/\xi)g(\gamma/\eta)K^2(\gamma)f_3(\gamma)\}d\gamma\,. \tag{2.39}$$

The functions g and f_1, f_2, f_3, Δ appearing in (2.39) are those previously defined in (2.27) and (2.31), respectively. Equation (2.38) is a Fredholm equation of the second kind for the weight-function D.

3. Displacements and stresses of the solution to the residual problem. Numerical evaluations and results.

The stress functions (2.18) involve the four weight-functions A, B, C, D and (2.29) relate A, B, C to D. Accordingly, (2.2), (2.3), (2.4) permit us to express the displacements and stresses of the solution $\overline{S}$ to the residual problem in terms of the solution D of the integral equation (2.38) and known functions. We now record the results reached in this manner and for this purpose adopt the additional notation

$$\left.\begin{aligned} \Omega_2'(\gamma,\rho) &= Y_2'(\gamma)J_2'(\gamma\rho)-J_2'(\gamma)Y_2'(\gamma\rho)\,,\\ G(\gamma,\rho) &= \frac{K_2(\gamma\rho)}{K_2(\gamma)}\;,\quad G'(\gamma,\rho) = \frac{\rho K_2'(\gamma\rho)}{K_2'(\gamma)}\,. \end{aligned}\right. \tag{3.1}$$

<u>Displacements of Solution</u> $\overline{S}$:

$$\left.\begin{aligned} \frac{v_r(\rho,\theta,\zeta)}{\cos 2\theta} &= \int_0^\infty D(\eta)[(1-\alpha+\eta\zeta)\eta^3\Omega_2'(\eta,\rho)\exp(-\eta\zeta)+w_r(\eta,\rho,\zeta)]d\eta\,,\\ \frac{v_\theta(\rho,\theta,\zeta)}{\sin 2\theta} &= \int_0^\infty D(\eta)[2(\alpha-1-\eta\zeta)\frac{\eta^2}{\rho}\Omega_2(\eta,\rho)\exp(-\eta\zeta)+w_\theta(\eta,\rho,\zeta)]d\eta\,,\\ \frac{v_z(\rho,\theta,\zeta)}{\cos 2\theta} &= \int_0^\infty D(\eta)[-(\alpha+\eta\zeta)\eta^3\Omega_2(\eta,\rho)\exp(-\eta\zeta)+w_z(\eta,\rho,\zeta)]d\eta\,, \end{aligned}\right\} \tag{3.2}$$

where

$$
\left.\begin{aligned}
w_r(\eta,\rho,\zeta) &= \frac{8}{\pi^2}\int_0^\infty \frac{\cos(\gamma\zeta)}{\rho\Delta(\gamma)}\Big[2\alpha g(\gamma/\eta)\{G(\gamma,\rho)[(\gamma^2\rho^2+4)f_2(\gamma)-2f_1(\gamma)] \\
&\quad +G'(\gamma,\rho)K(\gamma)f_4(\gamma)\}+g^2(\gamma/\eta)K(\gamma)\{G(\gamma,\rho)[(\gamma^2\rho^2+4)f_3(\gamma) \\
&\quad -8\alpha f_2(\gamma)]+G'(\gamma,\rho)[-(\gamma^2+4+\alpha K(\gamma))f_3(\gamma)+4\alpha f_2(\gamma)]\}\Big]d\gamma\ , \\
w_\theta(\eta,\rho,\zeta) &= \frac{16}{\pi^2}\int_0^\infty \frac{\cos(\gamma\zeta)}{\rho\Delta(\gamma)}\Big[\alpha g(\gamma/\eta)\{-2G(\gamma,\rho)f_4(\gamma)+G'(\gamma,\rho)K(\gamma)[f_1(\gamma) \\
&\quad -2f_2(\gamma)]\}+g^2(\gamma/\eta)\{G(\gamma,\rho)[(\gamma^2+4+\alpha K(\gamma))f_3(\gamma)-4\alpha f_2(\gamma)] \\
&\quad +G'(\gamma,\rho)K^2(\gamma)[-f_3(\gamma)+2\alpha f_2(\gamma)]\}\Big]d\gamma\ , \\
w_z(\eta,\rho,\zeta) &= \frac{8}{\pi^2}\int_0^\infty \frac{\gamma\sin(\gamma\zeta)}{\Delta(\gamma)}\Big[-2\alpha g(\gamma/\eta)\{G(\gamma,\rho)[f_4(\gamma)+2\alpha f_2(\gamma)] \\
&\quad +G'(\gamma,\rho)K(\gamma)f_2(\gamma)\}+g^2(\gamma/\eta)\{G(\gamma,\rho)[(\gamma^2+4-\alpha K(\gamma))f_3(\gamma) \\
&\quad -4\alpha f_2(\gamma)]-G'(\gamma,\rho)K^2(\gamma)f_3(\gamma)\}\Big]d\gamma\ .
\end{aligned}\right\} \quad (3.3)
$$

<u>Stresses</u> <u>of</u> <u>Solution</u> $\overline{S}$:

$$
\left.\begin{aligned}
\frac{\tau_{rr}(\rho,\theta,\zeta)}{\cos 2\theta} &= \int_0^\infty D(\eta)\Big[\{[\frac{4}{\rho^2}(1-\alpha+\eta\zeta)+(1-\eta\zeta)\eta^2]\Omega_2(\eta,\rho) \\
&\quad +(\alpha-1-\eta\zeta)\frac{\eta}{\rho}\Omega_2'(\eta,\rho)\}\eta^2\exp(-\eta\zeta)+t_{rr}(\eta,\rho,\zeta)\Big]d\eta\ , \\
\frac{\tau_{\theta\theta}(\rho,\theta,\zeta)}{\cos 2\theta} &= \int_0^\infty D(\eta)\Big[\{[\frac{4}{\rho^2}(\alpha-1-\eta\zeta)+(2-\alpha)\eta^2]\Omega_2(\eta,\rho) \\
&\quad +(1-\alpha+\eta\zeta)\frac{\eta}{\rho}\Omega_2'(\eta,\rho)\}\eta^2\exp(-\eta\zeta)+t_{\theta\theta}(\eta,\rho,\zeta)\Big]d\eta\ ,
\end{aligned}\right\} \quad (3.4)
$$

$$\left.\begin{aligned}
\frac{\tau_{zz}(\rho,\theta,\zeta)}{\cos 2\theta} &= \int_0^\infty D(\eta)[(1+\eta\zeta)\eta^4\Omega_2(\eta,\rho)\exp(-\eta\zeta)+t_{zz}(\eta,\rho,\zeta)]d\eta\ ,\\
\frac{\tau_{\theta z}(\rho,\theta,\zeta)}{\sin 2\theta} &= \int_0^\infty D(\eta)[\frac{2\eta^4\zeta}{\rho}\,\Omega_2(\eta,\rho)\exp(-\eta\zeta)+t_{\theta z}(\eta,\rho,\zeta)]\,d\eta\ ,\\
\frac{\tau_{zr}(\rho,\theta,\zeta)}{\cos 2\theta} &= \int_0^\infty D(\eta)[-\eta^5\zeta\Omega_2'(\eta,\rho)\exp(-\eta\zeta)+t_{zr}(\eta,\rho,\zeta)]\,d\eta\ ,\\
\frac{\tau_{r\theta}(\rho,\theta,\zeta)}{\sin 2\theta} &= \int_0^\infty D(\eta)\{2(1-\alpha+\eta\zeta)[\frac{1}{\rho^2}\Omega_2(\eta,\rho)\\
&\quad -\frac{\eta}{\rho}\Omega_2'(\eta,\rho)]\eta^2\exp(-\eta\zeta)+t_{r\theta}(\eta,\rho,\zeta)\}\,d\eta\ ,
\end{aligned}\right\}\text{(3. 4) cont.}$$

where

$$\left.\begin{aligned}
t_{rr}(\eta,\rho,\zeta) &= \frac{8}{\pi^2}\int_0^\infty \frac{\cos(\gamma\zeta)}{\rho^2\Delta(\gamma)}\Big\{2\alpha g(\gamma/\eta)\Big[G(\gamma,\rho)\{[(\alpha-1)\gamma^2\rho^2-4]f_2(\gamma)\\
&\quad +2f_1(\gamma)+(\gamma^2\rho^2+4)f_4(\gamma)\}+G'(\gamma,\rho)K(\gamma)[(\gamma^2\rho^2+4)f_2(\gamma)\\
&\quad -2f_1(\gamma)-f_4(\gamma)]\Big]+g^2(\gamma/\eta)\Big[G(\gamma,\rho)\{-[(\gamma^2\rho^2+4)(\gamma^2+4\\
&\quad +K(\gamma))+4\alpha K(\gamma)]f_3(\gamma)+4\alpha[\gamma^2\rho^2+4+2K(\gamma)]f_2(\gamma)\}\\
&\quad +G'(\gamma,\rho)K(\gamma)\{[(\gamma^2\rho^2+4+\alpha)K(\gamma)+\gamma^2+4]f_3(\gamma)\\
&\quad -4\alpha[2K(\gamma)+1]f_2(\gamma)\}\Big]\Big\}\,d\gamma\ ,\\
t_{\theta\theta}(\eta,\rho,\zeta) &= \frac{8}{\pi^2}\int_0^\infty \frac{\cos(\gamma\zeta)}{\rho^2\Delta(\gamma)}\Big\{2\alpha g(\gamma/\eta)\Big[G(\gamma,\rho)\{[(\alpha-1)\gamma^2\rho^2+4]f_2(\gamma)\\
&\quad -2f_1(\gamma)-4f_4(\gamma)\}+G'(\gamma,\rho)K(\gamma)[-4f_2(\gamma)+2f_1(\gamma)\\
&\quad +f_4(\gamma)]\Big]+g^2(\gamma/\eta)\Big[G(\gamma,\rho)\{[(\alpha-1)\gamma^2\rho^2K(\gamma)\\
&\quad +4(\gamma^2+4+K(\gamma)+\alpha K(\gamma))]f_3(\gamma)-8\alpha[K(\gamma)+2]f_2(\gamma)\}
\end{aligned}\right\}\text{(3. 5)}$$

$$
\left.
\begin{aligned}
&\qquad +G'(\gamma,\rho)K(\gamma)\{-[\gamma^2+4+4K(\gamma)+\alpha K(\gamma)]f_3(\gamma) \\
&\qquad +4\alpha[2K(\gamma)+1]f_2(\gamma)\}\Big]\Big\}d\gamma\ , \\[1ex]
t_{zz}(\eta,\rho,\zeta) &= \frac{8}{\pi^2}\int_0^\infty \frac{\gamma^2\cos(\gamma\zeta)}{\Delta(\gamma)}\Big[-2\alpha g(\gamma/\eta)\{G(\gamma,\rho)[(\alpha+2)f_2(\gamma) \\
&\qquad +f_4(\gamma)]+G'(\gamma,\rho)K(\gamma)f_2(\gamma)\}+g^2(\gamma/\eta)\{G(\gamma,\rho)[(\gamma^2+4 \\
&\qquad -2K(\gamma))f_3(\gamma)-4\alpha f_2(\gamma)]-G'(\gamma,\rho)K^2(\gamma)f_3(\gamma)\}\Big]d\gamma\ , \\[1ex]
t_{\theta z}(\eta,\rho,\zeta) &= \frac{8}{\pi^2}\int_0^\infty \frac{\gamma\sin(\gamma\zeta)}{\rho\Delta(\gamma)}\Big[\alpha g(\gamma/\eta)\{4G(\gamma,\rho)[\alpha f_2(\gamma)+f_4(\gamma)] \\
&\qquad +G'(\gamma,\rho)K(\gamma)[4f_2(\gamma)-f_1(\gamma)]\}+2g^2(\gamma/\eta)\{G(\gamma,\rho)[-(\gamma^2 \\
&\qquad +4)f_3(\gamma)+4\alpha f_2(\gamma)]+G'(\gamma,\rho)K^2(\gamma)[f_3(\gamma)-\alpha f_2(\gamma)]\}\Big]d\gamma\ , \\[1ex]
t_{zr}(\eta,\rho,\zeta) &= \frac{8}{\pi^2}\int_0^\infty \frac{\gamma\sin(\gamma\zeta)}{\rho\Delta(\gamma)}\Big[2\alpha g(\gamma/\eta)\{G(\gamma,\rho)[-(\gamma^2\rho^2+4)f_2(\gamma) \\
&\qquad +f_1(\gamma)]-G'(\gamma,\rho)K(\gamma)[\alpha f_2(\gamma)+f_4(\gamma)]\} \\
&\qquad +g^2(\gamma/\eta)K(\gamma)\{G(\gamma,\rho)[-(\gamma^2\rho^2+4)f_3(\gamma)+4\alpha f_2(\gamma)] \\
&\qquad +G'(\gamma,\rho)[(\gamma^2+4)f_3(\gamma)-4\alpha f_2(\gamma)]\}\Big]d\gamma\ , \\[1ex]
t_{r\theta}(\eta,\rho,\zeta) &= \frac{8}{\pi^2}\int_0^\infty \frac{\cos(\gamma\zeta)}{\rho^2\Delta(\gamma)}\Big\{\alpha g(\gamma/\eta)\{G(\gamma,\rho)[-4(\gamma^2\rho^2+4)f_2(\gamma) \\
&\qquad +(\gamma^2\rho^2+8)f_1(\gamma)+4f_4(\gamma)]+2G'(\gamma,\rho)K(\gamma)[2f_2(\gamma) \\
&\qquad -f_1(\gamma)-2f_4(\gamma)]\}+2g^2(\gamma/\eta)\Big[G(\gamma,\rho)\{-[(\gamma^2\rho^2+4 \\
&\qquad +\alpha)K(\gamma)+\gamma^2+4]f_3(\gamma)+\alpha[\gamma^2\rho^2K(\gamma)+4 \\
&\qquad +8K(\gamma)]f_2(\gamma)\}+G'(\gamma,\rho)K(\gamma)\{[\gamma^2+4+K(\gamma) \\
&\qquad +\alpha K(\gamma)]f_3(\gamma)-2\alpha[2+K(\gamma)]f_2(\gamma)\}\Big]\Big\}d\gamma\ .
\end{aligned}
\right\}\ \begin{matrix}(3.5)\\ \text{cont.}\end{matrix}
$$

For the sake of convenience we refer once more to the definitions of all auxiliary symbols appearing in $\overline{S}$: the functions G, G', Ω_2, Ω_2' are accounted for by (2.16), (3.1); K is given by (2.20) and f_1, f_2, f_3, f_4, Δ by (2.31); g and α are defined by (2.27) and (1.17), respectively. Note that $\overline{S}$ depends on Poisson's ratio in a complicated fashion. Thus, the parameter $\alpha \equiv 2(1-\nu)$ enters (3.2) to (3.5) not only explicitly but also through Δ, f_1, and f_4, as well as through D since the kernel (2.39) of (2.38) contains α, f_1, and Δ.

Once (2.38) has been solved for D, the desired displacements and stresses of the solution $\overline{S}$ to the residual problem are completely determined by (3.2) to (3.5). In view of the unwieldiness of the kernel (2.39) it was not feasible to treat (2.38) analytically. For this reason the integral equation (2.38) was solved numerically on an IBM-704 electronic computer for the values of Poisson's ratio $\nu = 1/4, 1/2$ ($\alpha = 3/2, 1$), the solution corresponding to $\nu = 0$ being known in advance.[20]

In order to keep the present paper to a reasonable length, the details of these and of subsequent numerical computations will have to be omitted here. A comprehensive account of the numerical work carried out, and of the extensive supplementary analytical work performed to accelerate the convergence of the solution, may be found in a separate report [21].

Since the derivation of $\overline{S}$ in Section 2 involves various purely formal manipulations whose validity depends upon the anticipated nature of the initially unknown weight-functions A, B, C, and D, some remarks concerning the aposteriori verification of the solution to the residual problem are in order. The numerical solution D of the Fredholm equation (2.38) is depicted in Figure 2. Further, a plot of $\log(D(\eta))$ versus $\log\eta$ indicates the asymptotic behavior

---

[20] Recall that $\overline{S}$ vanishes identically when $\nu = 0$.

$$D(\eta) = c(\nu)\eta^{-4} + o(\eta^{-4}) \quad \text{as } \eta \to \infty , \tag{3.6}$$

with $c(1/4) \doteq 0.322$, $c(1/2) \doteq 1.26$. On the other hand (3.6), together with the continuity of D on $[0, \infty)$, insures the required convergence of the improper integrals in (3.2) to (3.5) and entitles one to differentiate the displacements and stresses of $\overline{S}$ under the respective integral signs. Such differentiations, in turn, enable one to confirm that $\overline{S}$ indeed satisfies the cylindrical counterpart of the field equations (1.2), (1.3). Also, a tedious but straightforward computation based on (3.2) to (3.5) verifies that $\overline{S}$ meets all of the boundary conditions (1.26), the first two of (1.27), as well as (1.28), for every D that possesses the foregoing regularity properties. Finally, for every such D, the displacements (3.2) and the stresses (3.4) are found to vanish in the limit as $\zeta \to \infty$, so that, by (1.25), the solution S'' for Case 2 approaches the associated plane-strain solution $\mathring{S}''$ in this limit.

We have yet to confirm the last of the boundary conditions (1.27), i.e.,

$$\tau_{zz}(\rho, \theta, 0) = \frac{2(2-\alpha)}{\rho^2}\cos 2\theta \quad (1 \le \rho < \infty \ , \ 0 \le \theta < 2\pi) , \tag{3.7}$$

the fulfillment of which depends evidently upon the specific values of D. Accordingly, the verification of (3.7) supplies an essential check on the accuracy of the numerical solution of (2.38) and thereby gives an indication of the accuracy to be expected of the entire solution to the problem under consideration. The typical difficulties encountered in the numerical evaluation of the stress field appropriate to $\overline{S}$ are illustrated in especially severe form by those attending the computation of $\tau_{zz}(\rho, \theta, 0)$. To convey an idea of these complications, we first recall from (3.4) that

$$\frac{\tau_{zz}(\rho,\theta,\zeta)}{\cos 2\theta} = \int_0^\infty D(\eta)(1+\eta\zeta)\eta^4\Omega_2(\eta,\rho)\exp(-\eta\zeta)d\eta + \int_0^\infty D(\eta)t_{zz}(\eta,\rho,\zeta)d\eta\,, \tag{3.8}$$

where t_{zz} is itself an improper integral and is given by the third of (3.5). The first integrand in the right-hand member of (3.8), because of the factor $\exp(-\eta\zeta)$, decays rapidly as $\eta \to \infty$ when $\zeta > 0$. For $\zeta = 0$, however, this integrand is an oscillatory function of slowly decreasing amplitude. The improper integral representing t_{zz} is also found to be poorly convergent. Since an accurate knowledge of $t_{zz}(\eta, \rho, 0)$ is an essential prerequisite for the performance of the second integration required by (3.8), it was necessary to examine the asymptotic behavior of the integrand of t_{zz} for large values of η and to remove in closed form--in terms of sine and cosine integrals--certain contributions to $t_{zz}(\eta, \rho, 0)$ that impede its direct evaluation. An additional difficulty arises from the fact that the integral representation (3.8) for τ_{zz} is discontinuous along the edge $\rho = 1$, $\zeta = 0$. Indeed, one finds from (3.8) that

$$\tau_{zz}(1+, \theta, 0+) = \tau_{zz}(1, \theta, 0) + 4c(\nu)/\pi\,. \tag{3.9}$$

This discontinuous behavior is reflected in the slow convergence of (3.8) near $\rho = 1$, $\zeta = 0$. Nevertheless, as is apparent from Table 1, the error inherent in the computed values of $\tau_{zz}(\rho, \theta, 0)$ is insignificant also in the vicinity of $\rho = 1$. It will be observed, however, that the deviations of the numerical from the theoretical values of $\tau_{zz}(\rho, \theta, 0)$ increase in magnitude as ρ approaches unity. Finally, it should be emphasized that, for reasons already mentioned, the numerical results for the stresses away from the plane boundary, i.e., at $\zeta > 0$, are apt to be appreciably more accurate than those summarized in Table 1.

Of primary physical concern is the variation with ζ of the non-vanishing stresses along the boundary of the hole $\rho = 1$ and the radial variation of the normal displacement along the plane boundary $\zeta = 0$. A detailed account of the numerical evaluation of $\tau_{zz}(1, \theta, \zeta)$, $\tau_{\theta z}(1, \theta, \zeta)$, $\tau_{\theta\theta}(1, \theta, \zeta)$, and $u_z(\rho, \theta, 0)$ is included in [21]. The results obtained are plotted in Figures 3, 4, 5, 6 for Case 2, which corresponds to the state of pure shear (1.11) at infinity and represents the basic non-trivial loading case. Thus the present numerical results are based on the solution

$$S'' = \overset{\circ}{S}{}'' + \overline{S} , \tag{3.10}$$

in which $\overset{\circ}{S}{}''$ is the plane-strain solution (1.24) associated with Case 2, whereas $\overline{S}$ is the solution of the residual problem given by (3.2) to (3.5). Analogous numerical results for the general loading conditions (1.6) are immediately deducible from those presented here by means of (1.12) and (1.22).

Each of the diagrams to be discussed includes three curves, corresponding to the values of Poisson's ratio $\nu = 1/2$, $\nu = 1/4$, and $\nu = 0$. In this connection we recall that $\overset{\circ}{S}{}''$ is the exact solution for Case 2 when $\nu = 0$, so that

$$S'' = \overset{\circ}{S}{}'' \text{ for } \nu = 0 . \tag{3.11}$$

Also, since all components of $\overline{S}$ tend to zero as $\zeta \to 0$, one has

$$S'' \to \overset{\circ}{S}'' \text{ as } \zeta \to \infty , \tag{3.12}$$

regardless of the particular value of Poisson's ratio. In view of the fact that the residual tractions $\tau_{zz}(\rho, \theta, 0)$ to which $\overset{\circ}{S}''$ gives rise on $\zeta = 0$ are self-equilibrated, the conclusion (3.12) confirms an expectation suggested by (3.10) and an intuitive appeal to Saint-Venant's principle.[21]

Figures 3, 4 show the ζ-dependence at the cylindrical boundary of the transverse normal stress τ_{zz} and the transverse shear stress $\tau_{\theta z}$, respectively. The variation with ζ of the circumferential normal stress $\tau_{\theta\theta}$ at $\rho = 1$ is given in Figure 5. These graphs display clearly the three-dimensional boundary-layer effect that constitutes the main objective of the present paper. As is apparent from Figure 3, when $\nu > 0$, τ_{zz} departs radically from its respective plane-strain values (dashed lines) in the vicinity of $\zeta = 0$ but is already virtually indistinguishable from τ_{zz} of solution $\overset{\circ}{S}''$ at $\zeta = 3$. Similarly, the shear stress $\tau_{\theta z}$, which vanishes identically in $\overset{\circ}{S}''$, according to Figure 4 attains its maximum magnitude at approximately $\zeta = 0.35$ and decays rapidly as ζ increases beyond this value; at $\zeta = 3$ the magnitude of $\tau_{\theta z}$ is less than three percent of the maximum magnitude of $\tau_{\theta\theta}$. The relevant departures of $\tau_{\theta\theta}$ from its plane-strain values are confined to an even thinner boundary layer. Thus Figure 5 reveals that for $\nu = 1/2$ the plane-strain solution overestimates the magnitude of $\tau_{\theta\theta}$ up to $\zeta \doteq 0.45$, the actual value of $|\tau_{\theta\theta}(1, \theta, 0)/\cos 2\theta|$ being 2.35 as compared to the value of four predicted by $\overset{\circ}{S}''$. In contrast, the magnitude of $\tau_{\theta\theta}$ in S'' is only slightly larger than in $\overset{\circ}{S}''$ for $\zeta > 0.45$ when $\nu = 1/2$. Finally, Figure 6 depicts the dependence upon ρ of the

[21] Note, however, that the plane region Π over which $\overset{\circ}{S}''$ violates the boundary condition $\tau_{zz}(\rho, \theta, 0) = 0$ is unbounded.

normal displacement u_z at the plane boundary $\zeta = 0$. This displacement component vanishes identically in the plane-strain solution. As was to be anticipated, all of the three-dimensional effects under discussion are highly sensitive to changes in Poisson's ratio and become more pronounced at larger values of this parameter.

Appendix

Proof of the modified form of Weber's integral theorem.

We now establish the integral identity (2.32), which is closely related to Weber's integral theorem. Weber's theorem may be stated as follows. *Let* f *be continuously differentiable on* $[0, \infty)$ *and suppose the integral*

$$\int_0^\infty \gamma f(\gamma)d\gamma \tag{A1}$$

is absolutely convergent. Then, for every $\eta > 0$ *and* $n = 0, 1, 2, \ldots,$

$$\int_1^\infty \int_0^\infty \gamma\rho f(\gamma)\Lambda_n(\gamma,\rho)\Lambda_n(\eta,\rho)d\gamma d\rho = [J_n^2(\eta)+Y_n^2(\eta)]f(\eta) \ , \tag{A2}$$

where

$$\Lambda_n(\gamma,\rho) = Y_n(\gamma)J_n(\gamma\rho)-J_n(\gamma)Y_n(\gamma\rho) \ . \tag{A3}$$

Formula (A2) is originally due to Weber [22] and was later on deduced rigorously by Watson [12].[22] The proof appearing in [12] rests on the subsequent lemma.[23] *Let* $\epsilon > 0$ *and* g *be continuously differentiable on* $[\epsilon, \infty)$. *Suppose the integral*

$$\int_\epsilon^\infty \sqrt{\gamma}\, g(\gamma)\, d\gamma \tag{A4}$$

is absolutely convergent. Then, for every $\eta > 0$,

---

[22] See [12], p. 468. In Watson's proof n is no longer restricted to be an integer. Further, in [12] the range of integration for the second integral in (A2) is $[\epsilon, \infty)$, with $\epsilon > 0$, rather than $[0, \infty)$. Formula (A2) follows from Watson's result by passing to the limit as $\epsilon \to 0$. See also Titchmarsh [19].

[23] See [12], pp. 464-469, for a proof of the lemma.

$$\lim_{\lambda \to \infty} \sqrt{\lambda} \int_{\epsilon}^{\infty} \gamma g(\gamma) C_n(\gamma\lambda) d\gamma = 0 \, , \tag{A5}$$

$$\lim_{\lambda \to \infty} \int_{\epsilon}^{\infty} g(\gamma)[\gamma C_{n+1}(\gamma\lambda) C_n(\eta\lambda) - \eta C_{n+1}(\eta\lambda) C_n(\gamma\lambda)] \frac{\lambda \gamma d\gamma}{\gamma^2 - \eta^2} = \sigma^2 g(\eta) \, , \tag{A6}$$

<u>where</u> C_n <u>is the cylinder function defined by</u>

$$C_n(z) = \sigma[\cos\varphi J_n(z) + \sin\varphi Y_n(z)] \, , \tag{A7}$$

<u>in which</u> σ <u>and</u> φ <u>are arbitrary real constants.</u>

We state next the modification of Weber's theorem that is our present objective. <u>Let</u> f <u>satisfy the same hypotheses as in Weber's theorem. Then, for every</u> $\eta > 0$ <u>and</u> $n = 0, 1, 2, \ldots$,

$$\int_1^{\infty} \int_0^{\infty} \gamma \rho f(\gamma) \Omega_n(\gamma, \rho) \Omega_n(\eta, \rho) d\gamma d\rho = \{[J_n'(\eta)]^2 + [Y_n'(\eta)]^2\} f(\eta) \, , \tag{A8}$$

<u>where</u>

$$\Omega_n(\gamma, \rho) = Y_n'(\gamma) J_n(\gamma\rho) - J_n'(\gamma) Y_n(\gamma\rho) \, . \tag{A9}$$

As has been mentioned already, for the special case in which $n = 0$ a proof of the integral representation (A8) was given by Blenkarn [20]. We now adapt the argument used by Watson in dealing with (A2) to confirm (A8) for arbitrary non-negative integral values of n. To this end, let Ω_n' and M be the auxiliary functions defined by

$$\Omega_n'(\gamma, \rho) = Y_n'(\gamma) J_n'(\gamma\rho) - J_n'(\gamma) Y_n'(\gamma\rho) \, , \tag{A10}$$

$$M(\gamma, \eta, \rho) = \eta\rho \Omega_n(\gamma, \rho) \Omega_n'(\eta, \rho) - \gamma\rho \Omega_n(\eta, \rho) \Omega_n'(\gamma, \rho) \, . \tag{A11}$$

From (A9), (A10), (A11) follows

$$M(\gamma, \eta, 1) = 0 \tag{A12}$$

and further, with the aid of the recurrence relations for Bessel functions,

$$\frac{\partial}{\partial \rho} M(\gamma, \eta, \rho) = (\gamma^2 - \eta^2)\rho\Omega_n(\gamma, \rho)\Omega_n(\eta, \rho) . \tag{A13}$$

Consequently,

$$M(\gamma, \eta, \lambda) = (\gamma^2 - \eta^2)\int_1^\lambda \rho\Omega_n(\gamma, \rho)\Omega_n(\eta, \rho)d\rho . \tag{A14}$$

Define a function I on $(0, \infty)$ through

$$I(\eta) = \lim_{\lambda \to \infty} \int_\epsilon^\infty \gamma f(\gamma) \int_1^\lambda \rho\Omega_n(\gamma, \rho)\Omega_n(\eta, \rho)d\rho d\gamma \tag{A15}$$

so that, in view of (A14),

$$I(\eta) = \lim_{\lambda \to \infty} \int_\epsilon^\infty \frac{\gamma f(\gamma) M(\gamma, \eta, \lambda)}{\gamma^2 - \eta^2} d\gamma \tag{A16}$$

By virtue of (A11) and (A9), (A10), one can express M in (A16) in terms of Bessel functions of the first and second kinds and their first derivatives. Doing so, and invoking once again the Bessel recurrence relations, one finds, on setting

$$P_n(z) = J_n(z)+Y_n(z) , \; Q_n(z) = J_n(z)-Y_n(z) , \tag{A17}$$

that

$$I(\eta) = \sum_{j=1}^{5} I_j(\eta) \tag{A18}$$

where

$$\left.\begin{aligned}
I_1(\eta) &= \lim_{\lambda\to\infty}\int_\epsilon^\infty f(\gamma)Y_n'(\gamma)Y_n'(\eta)[\gamma J_{n+1}(\gamma\lambda)J_n(\eta\lambda) \\
&\qquad -\eta J_{n+1}(\eta\lambda)J_n(\gamma\lambda)]\frac{\lambda\gamma d\gamma}{\gamma^2-\eta^2}\,, \\
I_2(\eta) &= \lim_{\lambda\to\infty}\int_\epsilon^\infty f(\gamma)J_n'(\gamma)J_n'(\eta)[\gamma Y_{n+1}(\gamma\lambda)Y_n(\eta\lambda) \\
&\qquad -\eta Y_{n+1}(\eta\lambda)Y_n(\gamma\lambda)]\frac{\lambda\gamma d\gamma}{\gamma^2-\eta^2}\,, \\
I_3(\eta) &= -\lim_{\lambda\to\infty}\int_\epsilon^\infty \frac{1}{4}f(\gamma)[J_n'(\gamma)Y_n'(\eta)+J_n'(\eta)Y_n'(\gamma)][\gamma P_{n+1}(\gamma\lambda)P_n(\eta\lambda) \\
&\qquad -\eta P_{n+1}(\eta\lambda)P_n(\gamma\lambda)]\frac{\lambda\gamma d\gamma}{\gamma^2-\eta^2}\,, \\
I_4(\eta) &= \lim_{\lambda\to\infty}\int_\epsilon^\infty \frac{1}{4}f(\gamma)[J_n'(\gamma)Y_n'(\eta)+J_n'(\eta)Y_n'(\gamma)][\gamma Q_{n+1}(\gamma\lambda)Q_n(\eta\lambda) \\
&\qquad -\eta Q_{n+1}(\eta\lambda)Q_n(\gamma\lambda)]\frac{\lambda\gamma d\gamma}{\gamma^2-\eta^2}\,, \\
I_5(\eta) &= \lim_{\lambda\to\infty}\int_\epsilon^\infty \frac{1}{2}f(\gamma)[J_n'(\gamma)Y_n'(\eta)-J_n'(\eta)Y_n'(\gamma)][\gamma J_{n+1}(\gamma\lambda)Y_n(\eta\lambda) \\
&\qquad -\gamma Y_{n+1}(\gamma\lambda)J_n(\eta\lambda)+\eta J_{n+1}(\eta\lambda)Y_n(\gamma\lambda) \\
&\qquad -\eta Y_{n+1}(\eta\lambda)J_n(\gamma\lambda)]\frac{\lambda\gamma d\gamma}{\gamma^2-\eta^2}\,.
\end{aligned}\right\}\quad (A19)$$

We now apply[24] the lemma cited earlier to (A19) and conclude from (A6) that

$$I_5(\eta) = 0\,, \qquad (A20)$$

while inferring from (A5) that

---

[24] See [21], Appendix C, for details.

$$\left.\begin{aligned} I_1(\eta) &= [Y_n'(\eta)]^2 f(\eta),\ I_2(\eta) = [J_n'(\eta)]^2 f(\eta) , \\ I_3(\eta) &= -\ J_n'(\eta) Y_n'(\eta) f(\eta) ,\ I_4(\eta) = J_n'(\eta) Y_n'(\eta) f(\eta) . \end{aligned}\right\} \quad (A21)$$

But (A18), (A20), (A21) imply

$$I(\eta) = \{[J_n'(\eta)]^2 + [Y_n'(\eta)]^2\} f(\eta) , \qquad (A22)$$

which, because of (A15), yields the identity

$$\int_1^\infty \int_\epsilon^\infty \gamma\rho f(\gamma)\Omega_n(\gamma, \rho)\Omega_n(\eta, \rho) d\gamma d\rho = \{[J_n'(\eta)]^2 + [Y_n'(\eta)]^2\} f(\eta) . \qquad (A23)$$

Finally, the required integral representation (A8) follows from (A23) upon passing to the limit as $\epsilon \to 0$.

We recall that (A8) was applied in Section 2 to the function f given by

$$f(\gamma) = \gamma^3 D(\gamma) \qquad (A24)$$

and note that according to the empirical estimate (3.6), the integral (A1) fails to exist in this instance. On the other hand, the proof of a suitably stronger version of the integral theorem just established would necessitate a considerably more elaborate argument. At the same time even such a stronger theorem would not suffice to justify the inevitably formal application of (A8) to the solution of the main problem of this paper, whose validity depends upon the aposteriori verification of the final results discussed in Section 3.

	$\sigma_{zz}(\rho, \theta, 0)/\sigma \cos 2\theta$			
	$\nu = 1/4$		$\nu = 1/2$	
ρ	Numerical Value	Theoretical Value	Numerical Value	Theoretical Value
1	1.00218	1.00000	2.01167	2.00000
1.02	0.95576	0.96117	1.89914	1.92234
1.1	0.82730	0.82645	1.65466	1.65289
1.2	0.69578	0.69444	1.39262	1.38889
1.4	0.51100	0.51020	1.02146	1.02041
1.6	0.39070	0.39063	0.78148	0.78125
1.8	0.30879	0.30864	0.61737	0.61728
2	0.25011	0.25000	0.49982	0.50000
4	0.06228	0.06250	0.12432	0.12500
6	0.02706	0.02778	0.05381	0.05556
8	0.01523	0.01563	0.03120	0.03125
10	0.00903	0.01000	0.02027	0.02000

Table 1. Check on the boundary condition for σ_{zz} in the residual problem.

References

[1] E. Reissner, On the calculation of three-dimensional corrections for the two-dimensional theory of plane stress, Proc. Fifteenth Semi-Annual Eastern Photoelasticity Conference, 1942.

[2] A. E. Green, The elastic equilibrium of isotropic plates and cylinders, Proc. Roy. Soc. London, Ser. A, 195 (1949), p. 533.

[3] E. L. Reiss and S. Locke, On the theory of plane stress, Quart. Appl. Math., 19 (1961), 3, p. 195.

[4] K. O. Friedrichs, The edge effect in the bending of plates, Reissner Anniversary Volume, Edwards, Ann Arbor, 1949.

[5] E. Sternberg and R. Muki, Note on the expansion in powers of Poisson's ratio of solutions in elastostatics, Arch. Rat. Mech. Anal., 3 (1959), 3, p. 229.

[6] A. E. Green and T. J. Willmore, Three-dimensional stress systems in isotropic plates II, Proc. Roy. Soc. London, Ser. A, 193 (1948), p. 229.

[7] G. Kirsch, Die Theorie der Elastizität und die Bedürfnisse der Festigkeitslehre, Zentralblatt Verein Deutscher Ingenieure, 42 (1898), p. 797.

[8] A. E. Green, Three-dimensional stress systems in isotropic plates I, Trans. Roy. Soc. London, Ser. A, 240 (1948), 825, p. 561.

[9] E. Sternberg and M. A. Sadowsky, Three-dimensional solution for the stress concentration around a circular hole in a plate, J. Appl. Mech., 16 (1949), 1, p. 27.

[10] J. B. Alblas, Theorie van de driedimensionale spanningstoestand in een doorboovde plaat, Dissertation, Tech. Hogeschool Delft, H. J. Paris, Amsterdam, 1957.

[11] E. L. Reiss, Extension of an infinite plate with a circular hole, J. Soc. Indust. Appl. Math., 11 (1963), 4, p. 840.

[12] G. N. Watson, A treatise on the theory of Bessel functions, Second Ed., Cambridge Univ. Press, Cambridge, 1948.

[13] K. A. Blenkarn and J. C. Wilhoit, Stresses due to a band of normal stress at the entrance of a circular hole, J. Appl. Mech., 29 (1962), 4, p. 647.

[14] M. A. Sadowsky and Eli Sternberg, Stress concentration around a tri-axial ellipsoidal cavity, J. Appl. Mech., 16 (1949), 2, p. 149.

[15] S. Timoshenko and J. N. Goodier, Theory of elasticity, Second Ed., McGraw-Hill, New York, 1951.

[16] Tables of integral transforms, Vol. II, Bateman Manuscript Project, McGraw-Hill, New York, 1954.

[17] I. N. Sneddon, Fourier transforms, McGraw-Hill, New York, 1951.

[18] Tables of integral transforms, Vol. I, Bateman Manuscript Project, McGraw-Hill, New York, 1954.

[19] E. C. Titchmarsh, Weber's integral theorem, Proc. London Math. Soc., Ser. 2, 22 (1923), p. 15.

[20] K. A. Blenkarn, Fourier integral solutions to radially symmetric elasticity problems, Dissertation, Rice Institute, Houston, Texas, 1960.

[21] C. K. Youngdahl and E. Sternberg, Three-dimensional stress concentration around a cylindrical hole in a semi-infinite elastic body, Research and Development Report, Argonne National Laboratory. In preparation.

[22] H. Weber, Ueber die Darstellung willkürlicher Functionen durch Bessel'sche Functionen, Mathematische Annalen, 6 (1873), p. 146.

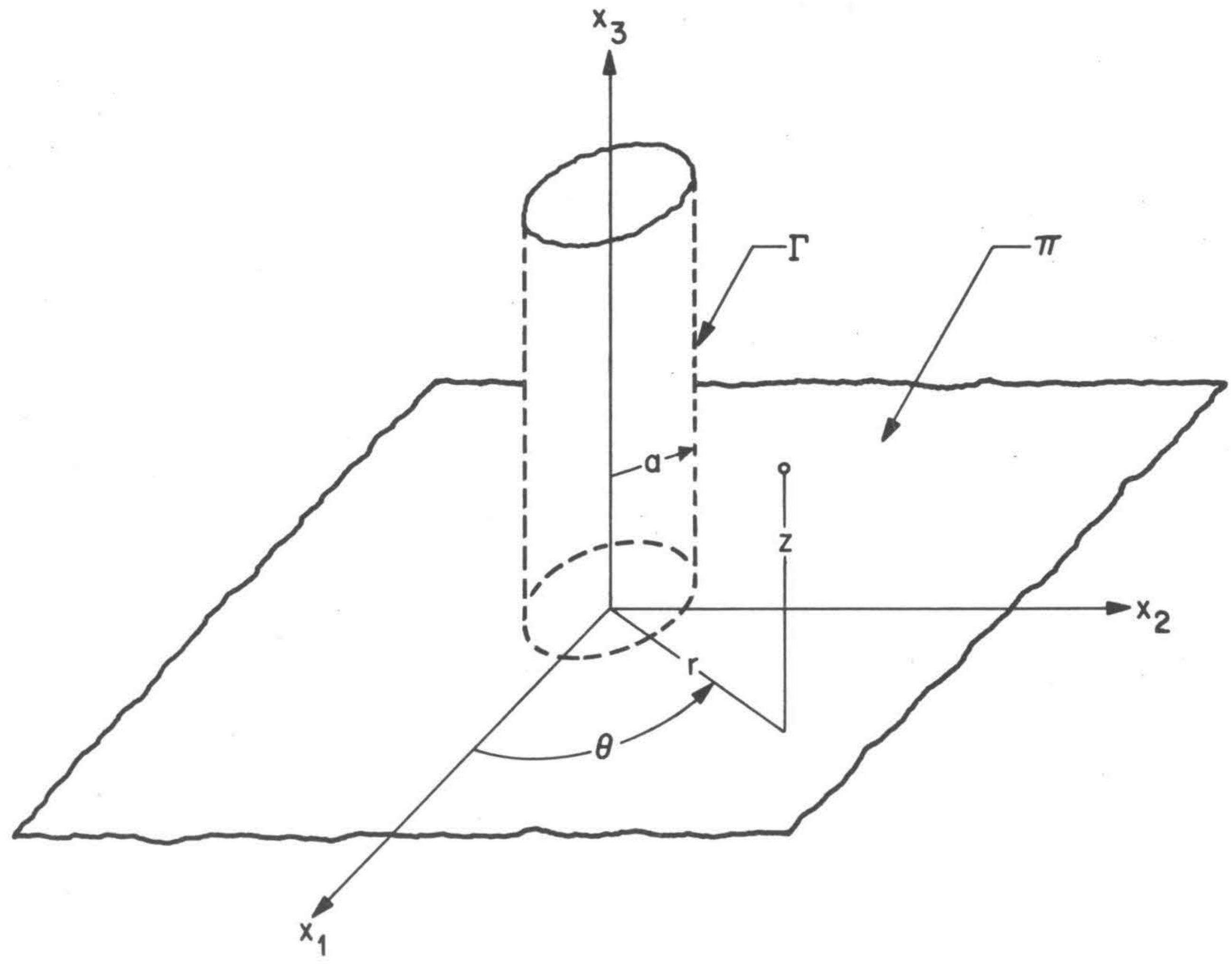

Figure 1. Half–space with cylindrical hole, cartesian and cylindrical coordinates.

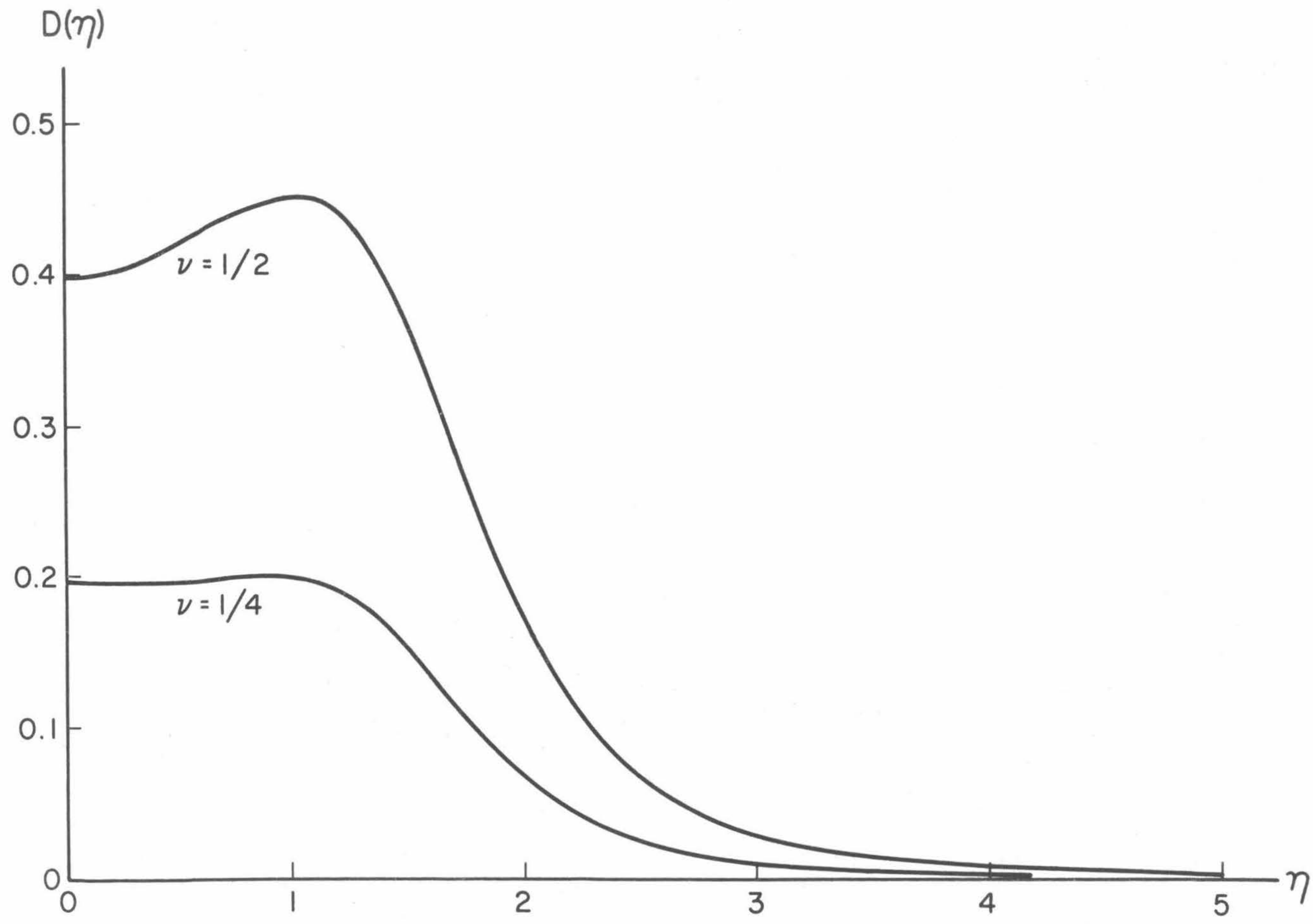

Figure 2. The solution of the integral equation for $D(\eta)$.

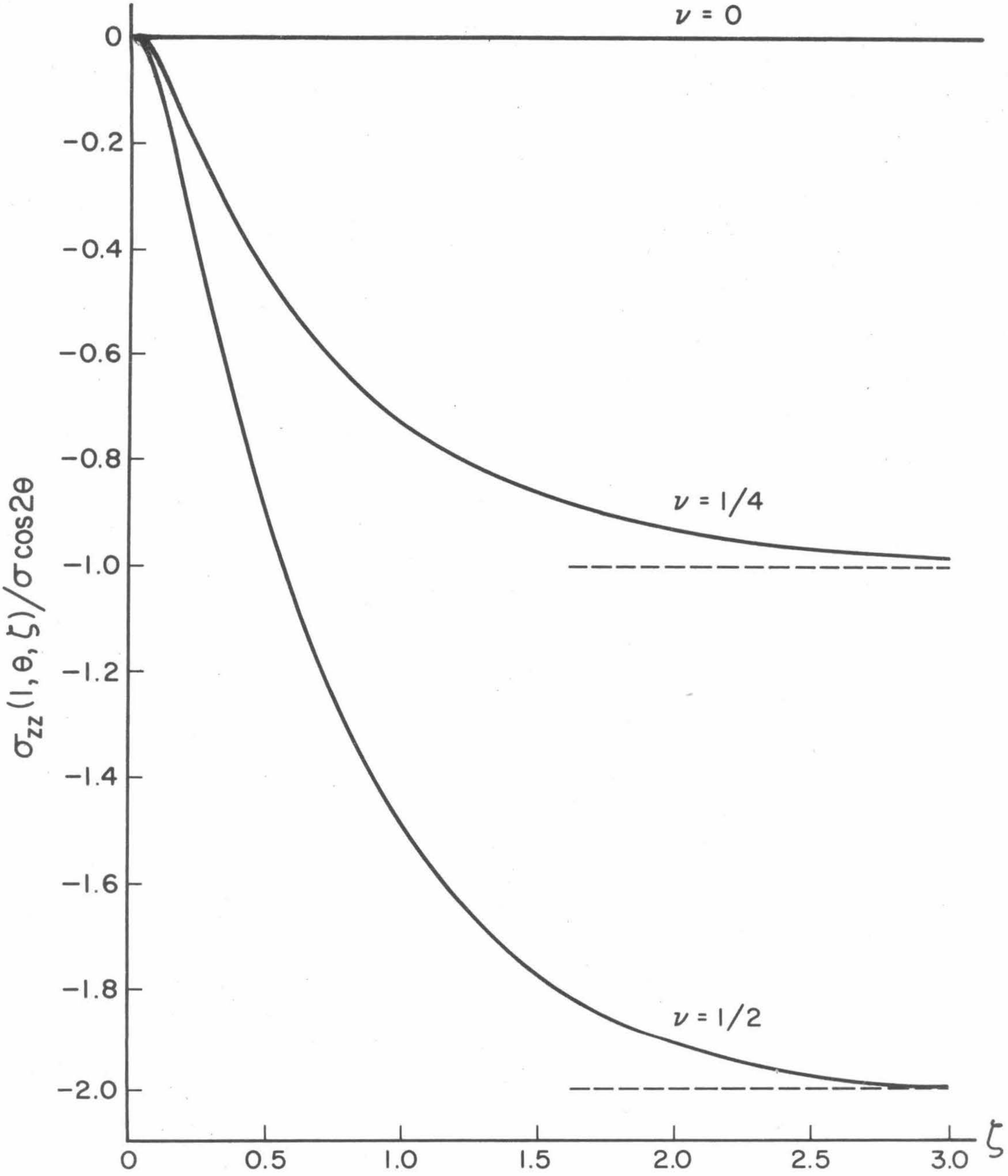

Figure 3. Dependence of σ_{zz} on ζ at $\rho = 1$

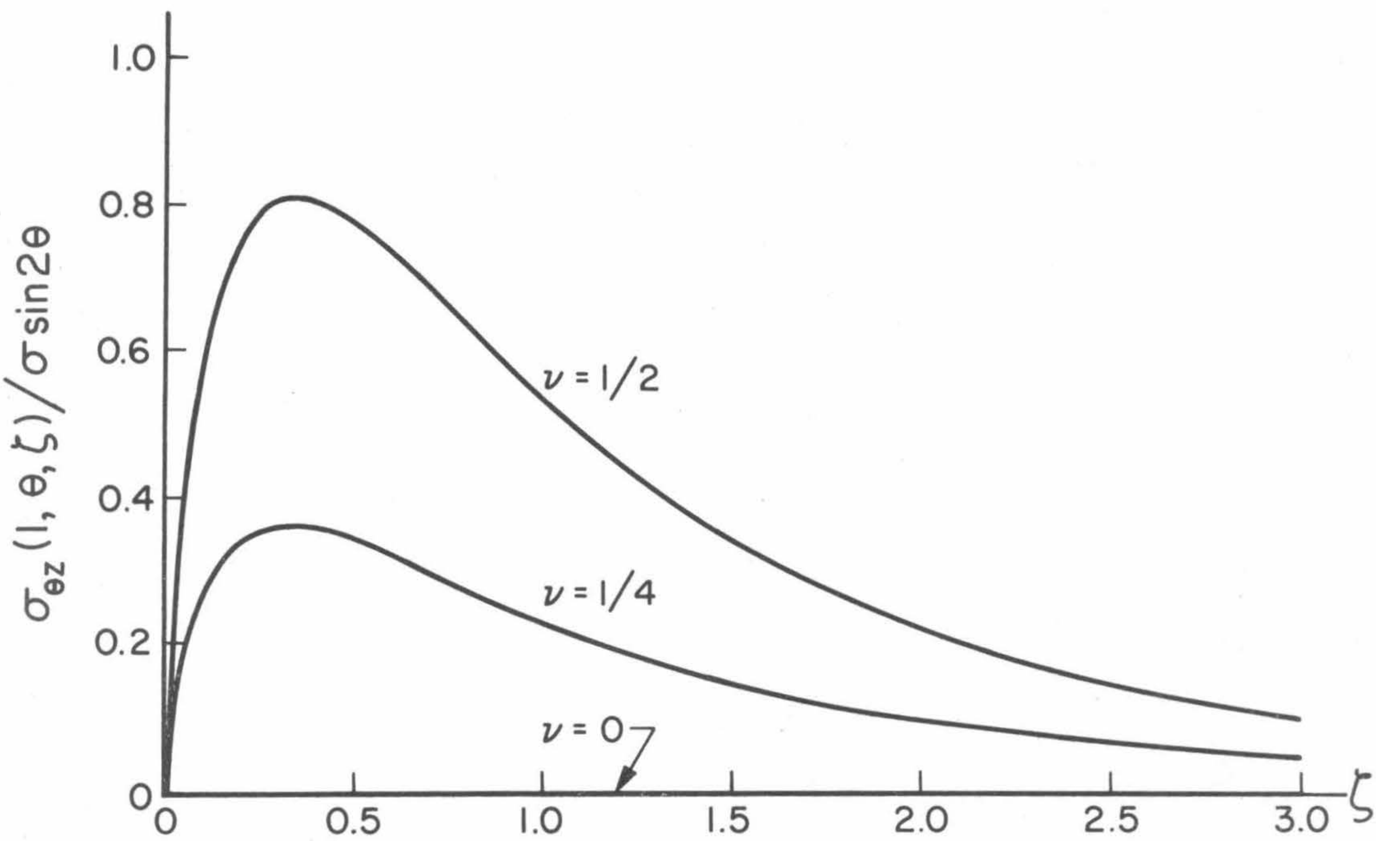

Figure 4. Dependence of $\sigma_{\theta z}$ on ζ at $\rho = 1$.

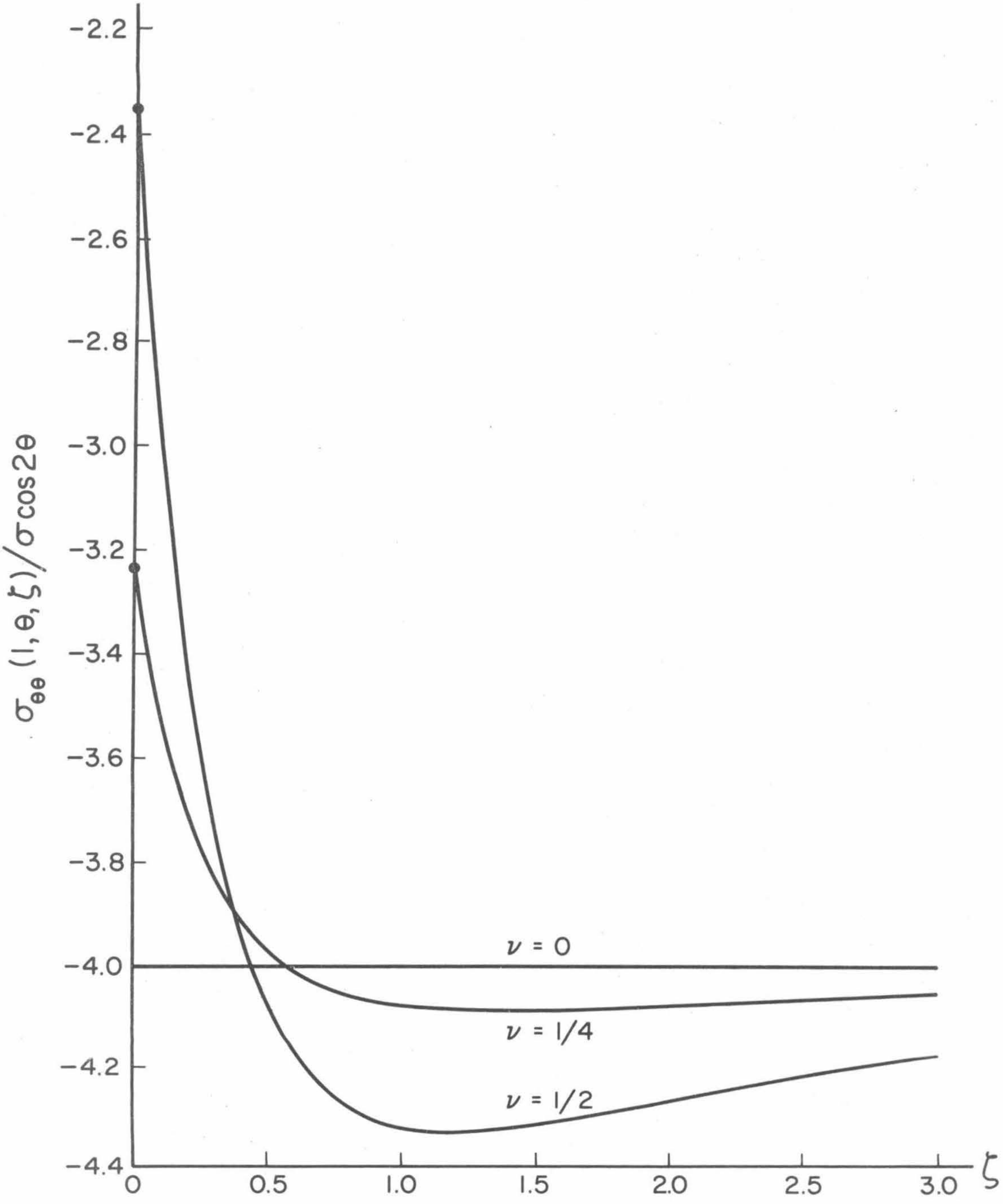

Figure 5. Dependence of $\sigma_{\theta\theta}$ on ζ at $\rho = 1$

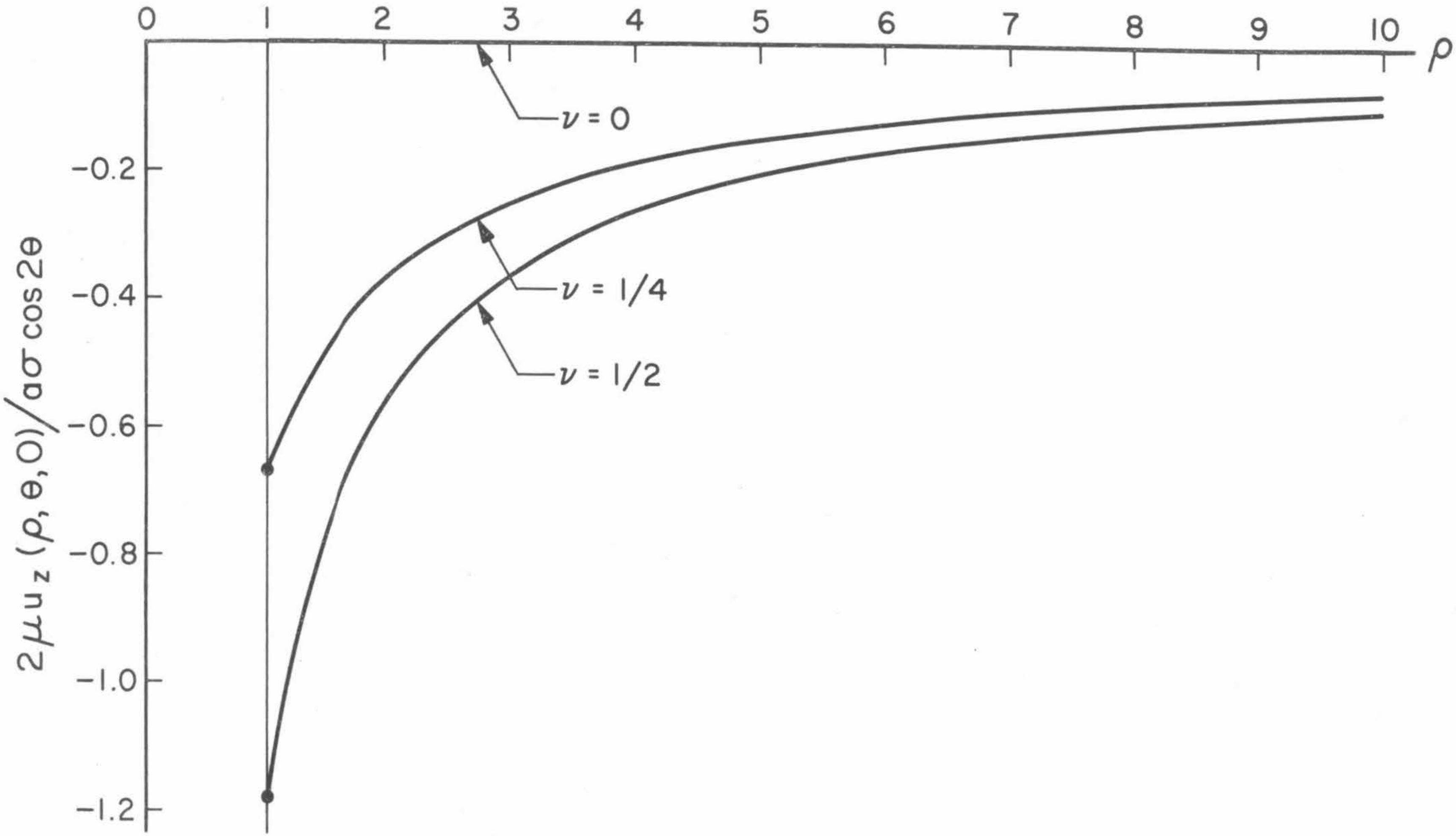

Figure 6. Dependence of u_z on ρ at $\zeta = 0$.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (*Corporate author*) California Institute of Technology		2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP
3. REPORT TITLE THREE-DIMENSIONAL STRESS CONCENTRATION AROUND A CYLINDRICAL HOLE IN A SEMI-INFINITE ELASTIC BODY		
4. DESCRIPTIVE NOTES (*Type of report and inclusive dates*)		
5. AUTHOR(S) (*Last name, first name, initial*) Youngdahl, Carl K. and Sternberg, Eli		
6. REPORT DATE May 1965	7a. TOTAL NO. OF PAGES 49	7b. NO. OF REFS 22
8a. CONTRACT OR GRANT NO. Nonr-220(58) b. PROJECT NO. c. NR-064-431 d.	9a. ORIGINATOR'S REPORT NUMBER(S) 2 9b. OTHER REPORT NO(S) (*Any other numbers that may be assigned this report*)	
10. AVAILABILITY/LIMITATION NOTICES		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research	

13. ABSTRACT

This paper contains a three-dimensional solution, exact within classical elastostatics, for the stresses and deformations arising in a halfspace with a semi-infinite transverse cylindrical hole, if the body--at infinite distances from its cylindrical boundary--is subjected to an arbitrary uniform plane field of stress that is parallel to the bounding plane. The solution presented is in integral form and is deduced with the aid of the Papkovich stress functions by means of an especially adapted, unconventional, integral-transform technique. Numerical results for the non-vanishing stresses along the boundary of the hole and for the normal displacement at the plane boundary, corresponding to several values of Poisson's ratio, are also included. These results exhibit in detail the three-dimensional stress boundary layer that emerges near the edges of the hole in the analogous problem for a plate of finite thickness, as the ratio of the plate-thickness to the diameter of the hole grows beyond bounds. The results obtained thus illustrate the limitations inherent in the two dimensional plane-strain treatment of the spatial plane problem; in addition, they are relevant to failure considerations and are of interest in connection with experimental stress analysis.